Topics in concentration of measure: Lecture III

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Lecture III: Large deviations for dense random graphs

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- What metric? What space?

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- Number of triangles in $G(n, p)$ roughly $\binom{n}{3}p^3 \sim n^3p^3/6$.
- What if, just by chance, #triangles turns out to be $\approx tn^3$ where $t > p^3/6$? What would the graph look like, conditional on this rare event?

An abstract topological space of graphs

- Beautiful unifying theory developed by Lovász and coauthors V. T. Sós, B. Szegedy, C. Borgs, J. Chayes, K. Vesztergombi, A. Schrijver and M. Freedman. Related to earlier works of Aldous, Hoover, Kallenberg.

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- Let G_n be a sequence of simple graphs whose number of nodes tends to infinity.
- For every fixed simple graph H, let $\hom(H, G)$ denote the number of homomorphisms of H into G (i.e. edge-preserving maps $V(H) \to V(G)$, where $V(H)$ and $V(G)$ are the vertex sets).

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- For every fixed simple graph H, let $\hom(H, G)$ denote the number of homomorphisms of H into G (i.e. edge-preserving maps $V(H) \to V(G)$, where $V(H)$ and $V(G)$ are the vertex sets).
- This number is normalized to get the homomorphism density

$$t(H, G) := \frac{\hom(H, G)}{|V(G)|^{|V(H)|}}.$$

This gives the probability that a random mapping $V(H) \to V(G)$ is a homomorphism.

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- Conversely, every such function arises as the limit of an appropriate graph sequence.
- This limit object determines all the limits of subgraph densities: if H is a simple graph with k vertices, then

$$t(H, f) = \int_{[0,1]^k} \prod_{(i,j)\in E(H)} f(x_i, x_j)\, dx_1 \cdots dx_k.$$

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- A sequence of graphs $\{G_n\}_{n\geq 1}$ is said to converge to f if for every finite simple graph H,

$$\lim_{n\to\infty} t(H, G_n) = t(H, f).$$

Example

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- On the other hand, if f is the function that is identically equal to p, then $t(H,f) = p^{|E(H)|}$.
- Thus, the sequence of random graphs $G(n,p)$ converges almost surely to the non-random limit function $f(x,y) \equiv p$ as $n \to \infty$.

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- So, what is the topology on this space?

The cut metric

- For any $f, g \in \mathcal{W}$, Frieze and Kannan defined the cut distance:

$$d_\square(f,g) := \sup_{S,T\subseteq[0,1]} \left| \int_{S\times T} [f(x,y) - g(x,y)]dxdy \right|.$$

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- The quotient space is denoted by $\widetilde{\mathcal{W}}$ and τ denotes the natural map $g \to \widetilde{g}$.
- Since $d_\square$ is invariant under σ one can define on $\widetilde{\mathcal{W}}$ the natural distance $\delta_\square$ by

$$\delta_\square(\widetilde{f}, \widetilde{g}) := \inf_\sigma d_\square(f, g_\sigma) = \inf_\sigma d_\square(f_\sigma, g) = \inf_{\sigma_1,\sigma_2} d_\square(f_{\sigma_1}, g_{\sigma_2})$$

making $(\widetilde{\mathcal{W}}, \delta_\square)$ into a metric space.

To any finite graph G, we associate the natural graphon f^G and its orbit $\widetilde{G} = \tau f^G = \widetilde{f}^G \in \widetilde{\mathcal{W}}$. One of the key results of the is the following:

Theorem (Borgs, Chayes, Lovász, Sós & Vesztergombi)

A sequence of graphs $\{G_n\}_{n\geq 1}$ converges to a limit $f \in \mathcal{W}$ if and only if $\delta_\Box(\widetilde{G}_n, \widetilde{f}) \to 0$ as $n \to \infty$.

Our result

- For any Borel set $\widetilde{A} \subseteq \widetilde{\mathcal{W}}$, let

$$\widetilde{A}_n := \{\widetilde{h} \in \widetilde{A} : \widetilde{h} = \widetilde{G} \text{ for some } G \text{ on } n \text{ vertices}\}.$$

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Theorem (Chatterjee & Varadhan, 2010)

The function I is well-defined and lower-semicontinuous on $\widetilde{\mathcal{W}}$. If $\widetilde{F}$ is a closed subset of $\widetilde{\mathcal{W}}$ then

$$\limsup_{n\to\infty} n^{-2} \log |\widetilde{F}_n| \le - \inf_{\widetilde{h}\in\widetilde{F}} I(\widetilde{h})$$

and if $\widetilde{U}$ is an open subset of $\widetilde{\mathcal{W}}$, then

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- Indeed, the main result in our paper is stated as a large deviation principle for the Erdős-Rényi graph, which can be easily proved to be equivalent to the graph counting principle stated before.

Large deviation principle for ER graphs

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Theorem (Chatterjee & Varadhan, 2010)

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- However, the weak topology is not very interesting. For example, subgraph counts are not continuous with respect to the weak topology.
- The LDP for the topology of the cut metric does not follow via standard methods.

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- Call a pair (A, B) of disjoint sets $A, B \subseteq V$ ϵ-regular if all $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$ satisfy $|\rho_G(X, Y) - \rho_G(A, B)| \leq \epsilon$.

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Theorem (Szemerédi's lemma)

Given $\epsilon > 0$, $m \geq 1$ there exists $M = M(\epsilon, m)$ such that every graph of order $\geq M$ admits an ϵ-regular partition $\{V_0, \ldots, V_K\}$ for some $K \in [m, M]$.

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- Using Szemerédi's regularity lemma, one can prove that $\delta_\square(G, G') \simeq 0$ with high probability if K and n are appropriately large and ϵ is small.
- Let f be the probability density of the law of $G(n, p)$ with respect to the law of G'. (This is easily computed; gives rise to the entropy function.)

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- Let G' be the random graph with independent edges where a vertex $u \in V_i$ is connected to a vertex $v \in V_j$ with probability $\rho_G(V_i, V_j)$.
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- Since the space $\widetilde{\mathcal{W}}$ is compact, this allows us to approximate $\mathbb{P}(G(n,p) \in A)$ for any nice set A by approximating A as a finite union of small balls.

Conditional distributions

Theorem

Take any $p \in (0,1)$. Let $\widetilde{F}$ be a closed subset of $\widetilde{\mathcal{W}}$ satisfying

$$\inf_{\widetilde{h} \in \widetilde{F}^o} I_p(\widetilde{h}) = \inf_{\widetilde{h} \in \widetilde{F}} I_p(\widetilde{h}) > 0.$$

Let $\widetilde{F}^*$ be the subset of $\widetilde{F}$ where I_p is minimized. Then $\widetilde{F}^*$ is non-empty and compact, and for each n, and each $\epsilon > 0$,

$$\mathbb{P}(\delta_\square(G(n,p), \widetilde{F}^*) \geq \epsilon \mid G(n,p) \in \widetilde{F}) \leq e^{-C(\epsilon, \widetilde{F}) n^2}$$

where $C(\epsilon, \widetilde{F})$ is a positive constant depending only on ϵ and $\widetilde{F}$.

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Proof: Follows from the compactness of $\widetilde{\mathcal{W}}$ (a deep result of Lovász and Szegedy, involving recursive applications of Szemerédi's lemma and martingales).

Large deviations for triangle counts

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- Exact evaluation of limit due to Chatterjee & Dey (2009): for a certain explicit set of (p,t),

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}(T_{n,p} \ge tn^3) = -I_p((6t)^{1/3}),$$

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- Unfortunately, the result does not cover all values of (p,t).

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Large deviations for triangle counts contd.

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$$T(f) := \frac{1}{6}\int_0^1\int_0^1\int_0^1 f(x,y)f(y,z)f(z,x)\, dx\, dy\, dz$$

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- For each $p \in (0,1)$ and $t \geq 0$, let

$$\phi(p,t) := \inf\{I_p(f) : f \in \mathcal{W},\ T(f) \geq t\}. \qquad (1)$$

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Theorem (Chatterjee & Varadhan, 2010)

For each $p \in (0,1)$ and each $t \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}(T_{n,p} \geq tn^3) = -\phi(p,t).$$

Moreover, the infimum is attained in the variational problem (1).

The ‘replica symmetric’ phase

Theorem (Chatterjee & Varadhan, 2010)

Let $h_p(t) := I_p((6t)^{1/3})$. Let $\hat{h}_p$ be the convex minorant of h_p. If t is a point where $h_p(t) = \hat{h}_p(t)$, then $\phi(p,t) = h_p(t)$. Moreover, for such (p,t), the conditional distribution of $G(n,p)$ given $T_{n,p} \geq tn^3$ is indistinguishable from the law of $G(n,(6t)^{1/3})$ in the large n limit.

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Remarks: This result recovers the result of Chatterjee & Dey and gives more. However, the theorem of Chatterjee & Dey gives an error bound of order $n^{-1/2}$, which is impossible to obtain via Szemerédi's lemma.

‘Replica symmetry breaking’

The following theorem shows that given any t, for all p small enough, the conditional distribution of $G(n, p)$ given $T_{n,p} \geq tn^3$ does not resemble that of an Erdős-Rényi graph.

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Theorem (Chatterjee & Varadhan, 2010)

Let $\widetilde{C}$ denote the set of constant functions in $\widetilde{\mathcal{W}}$ (representing all Erdős-Rényi graphs). For each t, there exists $p' > 0$ and $\epsilon > 0$ such that for all $p < p'$,

$$\lim_{n\to\infty} \mathbb{P}(\delta_{\square}(G(n,p), \widetilde{C}) > \epsilon \mid T_{n,p} \geq tn^3) = 1.$$

The double phase transition

Theorem (Chatterjee & Varadhan, 2010)

There exists $p_0 > 0$ such that if $p \leq p_0$, then there exists $p^3/6 < t' < t'' < 1/6$ such that the replica symmetric picture holds when $t \in (p^3/6, t') \cup (t'', 1/6)$, but there is a non-empty subset of (t', t'') where replica symmetry breaks down.

The small p limit

The following theorem says that when t is fixed and p is very small, then conditionally on the event $\{T_{n,p} \geq tn^3\}$ the graph $G(n,p)$ must look like a clique.

Theorem (Chatterjee & Varadhan, 2010)

For each t,

$$\lim_{p\to 0} \frac{\phi(p,t)}{\log(1/p)} = \frac{(6t)^{2/3}}{2}.$$

Moreover, if

$$\chi_t(x,y) := 1_{\{\max\{x,y\}\leq (6t)^{1/3}\}}$$

is the graphon representing a clique with triangle density t, then for each $\epsilon > 0$,

$$\lim_{p\to 0}\lim_{n\to\infty} \mathbb{P}(\delta_\square(\widetilde{G(n,p)}, \widetilde{\chi}_t) \geq \epsilon \mid T_{n,p} \geq tn^3) = 0.$$

Lower tails

- Given a fixed simple graph H,

$$\lim_{u\to 0}\lim_{n\to\infty}\frac{\log \mathbb{P}(t(H,G(n,p))\le u)}{n^2} = -\frac{1}{2(\chi(H)-1)}\log\frac{1}{1-p},$$

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- However, if $t(H,G(n,p))$ is just a little bit below its expected value, the graph continues to look like an Erdős-Rényi graph as in the upper tail case.

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- Gives interesting phase transitions, confirming predictions from the non-rigorous literature.

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Acknowledgment

Special thanks to: Amir Dembo, who suggested the problem to me in 2005. An old manuscript due to Bolthausen, Comets and Dembo (2003) provided a partial solution to the question but was never published.