Random nodal portraits

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June, 2012

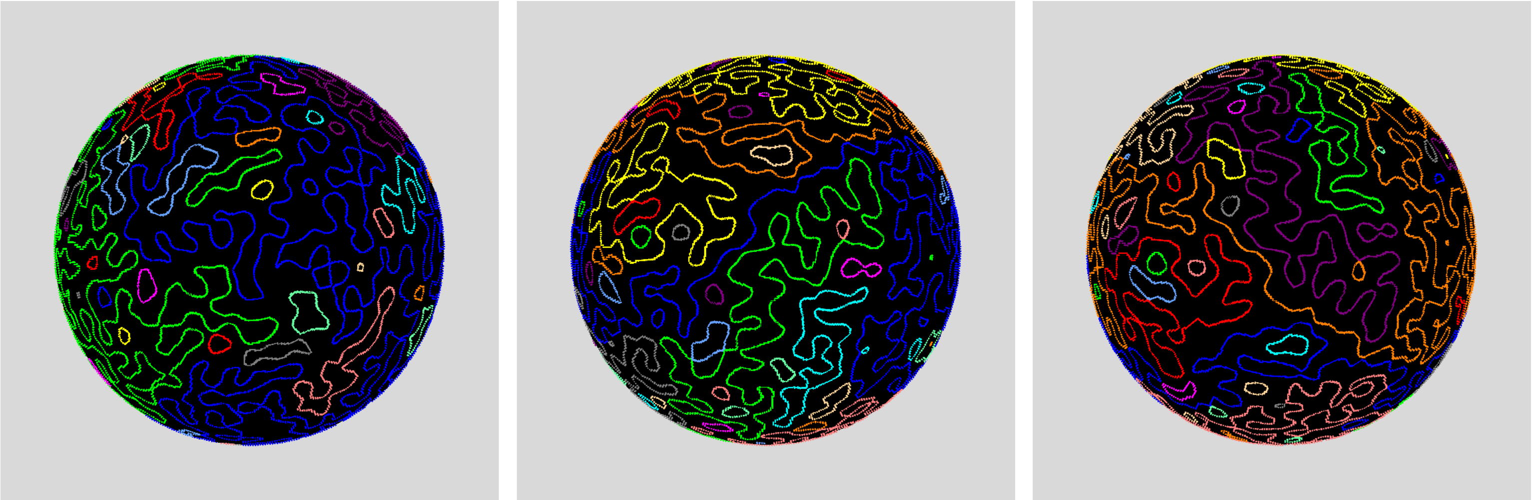
Statistical version of Hilbert's sixteenth problem

Hilbert's sixteenth problem:

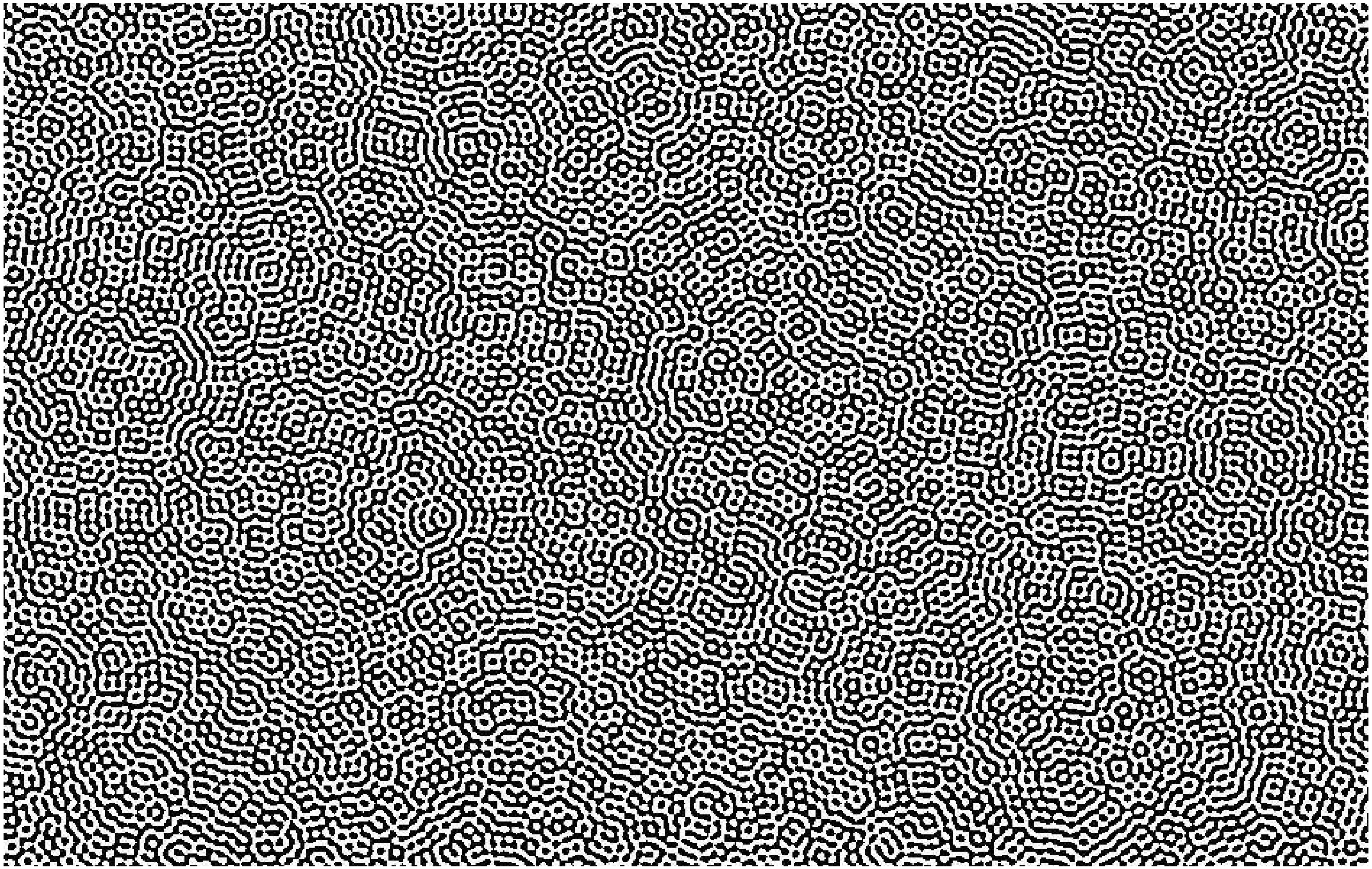
"The maximum number of closed and separate branches which a plane algebraic curve of the n-th order can have has been determined by Harnack. There arises the further question as to the relative position of the branches in the plane. ... A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space."

David Hilbert, International Congress of Mathematicians, Paris, 1900

Nodal lines of Kostlan ensemble of degree 56 on $\mathbb{S}^2$

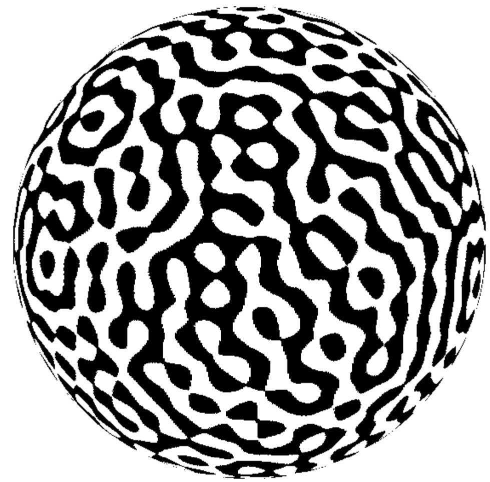


Nodal portraits created by Maria Nastasescu

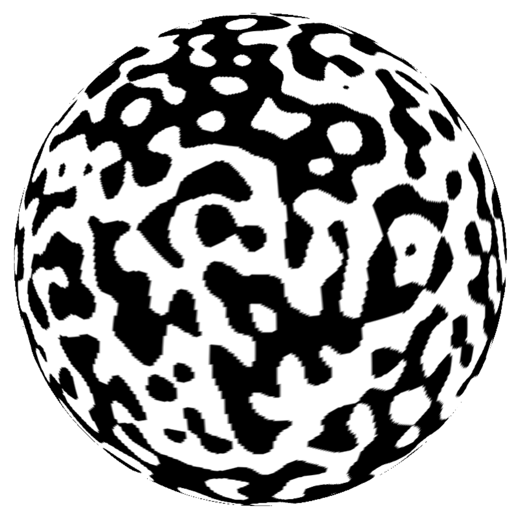


Nodal portrait of a gaussian plane wave (figure by Alex Barnett)

Gaussian spherical harmonic of degree 40



Gaussian linear combination of spherical harmonic of degrees ≤ 40



Nodal protraits created by Alex Barnett

Questions that we cannot answer:

Questions pertaining to $2D$ Gaussian monochromatic random waves

Challenge: Reveal "hidden universality law" that provides the rigorous foundation for the Bogomolny-Schmit work

Question: Show that a.s. there is no infinite nodal line

Question: Show that for each $\epsilon > 0$, the probability that the set $\{x\colon F(x) > \epsilon, |x| < R\}$ has a component with diameter $\geq \epsilon R$ tends to zero.

Question: Find asymptotics of the variance of $N(R; F)$ (according to the B-S prediction it grows as const $\cdot\, R^2$)

Similar questions can be raised for the spherical harmonics ensemble.

Questions that we cannot answer: (continuation)

Questions pertaining to high-energy Laplace eigenfunctions

Nothing is known about the number of connected components of the nodal set for 'randomly chosen' high-energy Laplace eigenfunction f_λ on an arbitrary compact surface M without boundary endowed with a smooth Riemannian metric g

Does Theorem 3 (on spherical harmonics) models what is happening when $M = \mathbb{S}^2$ and g is a generic metric sufficiently close to the constant one?

Let $\lambda_1 < \lambda_2 < \ldots$ be a (simple) spectrum of the Laplacian $-\Delta_g$ on M
$C \gg 1$ sufficiently big constant, $n \to \infty$
Consider f_λ's with $\sqrt{\lambda} \in \big[n, n+C\big]$, their number is $\simeq n$

Question: Whether at least for a given proportion of such eigenfunctions, $N(f_\lambda) \geq cn^2$?

Instead of perturbing the 'round metric' on $\mathbb{S}^2$, one can add a small (random) potential to the spherical Laplacian. The question remains hard

Questions that we cannot answer: (continuation)

Questions pertaining to Euclidean models

Question: Unveil the nature of the constant $\nu(\rho)$ in Theorem 1. At least, find tight bounds for $\nu(\rho)$

Question: Find statistics of large components of the zero set of F; i.e., components of diameter comparable to R^α with $0 < \alpha < 1$

Question: Find asymptotic of the variance of $N(R; F)$

Question: Prove exponential concentration of $N(R; F)/R^m$ around $\nu(\rho)$

The difficulty is caused by components of small diameter, which do not exist when F satisfies the Helmholtz equation $\Delta F + \kappa^2 F = 0$

Even for stationary Gaussian processes on $\mathbb{R}$, the question about exponential concentration remains open

Some refs:

E. Bogomolny, C. Schmit, Percolation Model for Nodal Domains of Chaotic Wave Functions. Phys. Rev. Letters, **88** (2002)

T. L. Malevich, Contours that arise when the zero level is crossed by Gaussian fields. Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **16** (1972), 20–23.

F. Nazarov, M. Sodin, On the number of nodal domains of random spherical harmonics. Amer. J. Math. **131** (2009)

F. Nazarov, M. Sodin, On the number of connected components of the nodal sets of smooth Gaussian functions. In preparation

P. Sarnak, Letter to B. Gross and J. Harris on ovals of random plane curves, May 2011. `http://publications.ias.edu/sarnak/paper/510`

B. Tsirelson, Gaussian measures. Lecture notes. Tel Aviv University, Fall 2010. `http://www.tau.ac.il/~tsirel/Courses/Gauss3/main.html`