Painlevé Transcendents

and their appearance in physics and random matrices

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Painlevé Equations

I. $u_{xx} = 6u^2 + x$

II. $u_{xx} = xu + 2u^3 + \alpha$

III. $u_{xx} = \frac{1}{u}{u_x}^2 - \frac{u_x}{x} + \frac{1}{x}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}$

IV. $u_{xx} = \frac{1}{2u}{u_x}^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u}$

V. $u_{xx} = \frac{3u-1}{2u(u-1)}{u_x}^2 - \frac{1}{x}u_x + \frac{(u-1)^2}{x^2}(\alpha u + \frac{\beta}{u}) + \frac{\gamma u}{x} + \frac{\delta u(u+1)}{u-1}$

VI. $$u_{xx} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right){u_x}^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right)u_x + \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right)$$

($\nu, \alpha, \beta, \gamma, \delta$ are complex parameters)

P. Painlevé B. Gambier (1900, 1910).

. Introduction.

$$u_{xx} = F(u, u_x, x) \oplus \text{P. P.}$$

$\Downarrow$ Painlevé, Gambier.

$$\boxed{\begin{array}{c}\text{Painlevé Equations} \\ P_1 - P_6\end{array}}$$

New transcendents

Umemura

Watanabe

$\Uparrow$

Isomonodromy deformations of

$$\frac{d\Psi}{d\lambda} = A(\lambda; x)\Psi$$

Garnier, Fuchs, Schlesinger

Jimbo, Miwa; Flaschka, Newell.

$\Downarrow$

$$\Downarrow$$

The Riemann-Hilbert representation of Painlevé transcendents:

$$Y_+(\lambda) = Y_-(\lambda)\, G(\lambda, x)$$

PE $\Leftrightarrow$ $$\lambda \in \Gamma$$

$$G(\lambda, x) = e^{i\theta(\lambda, x)}\, S\, e^{-i\theta(\lambda, x)}$$

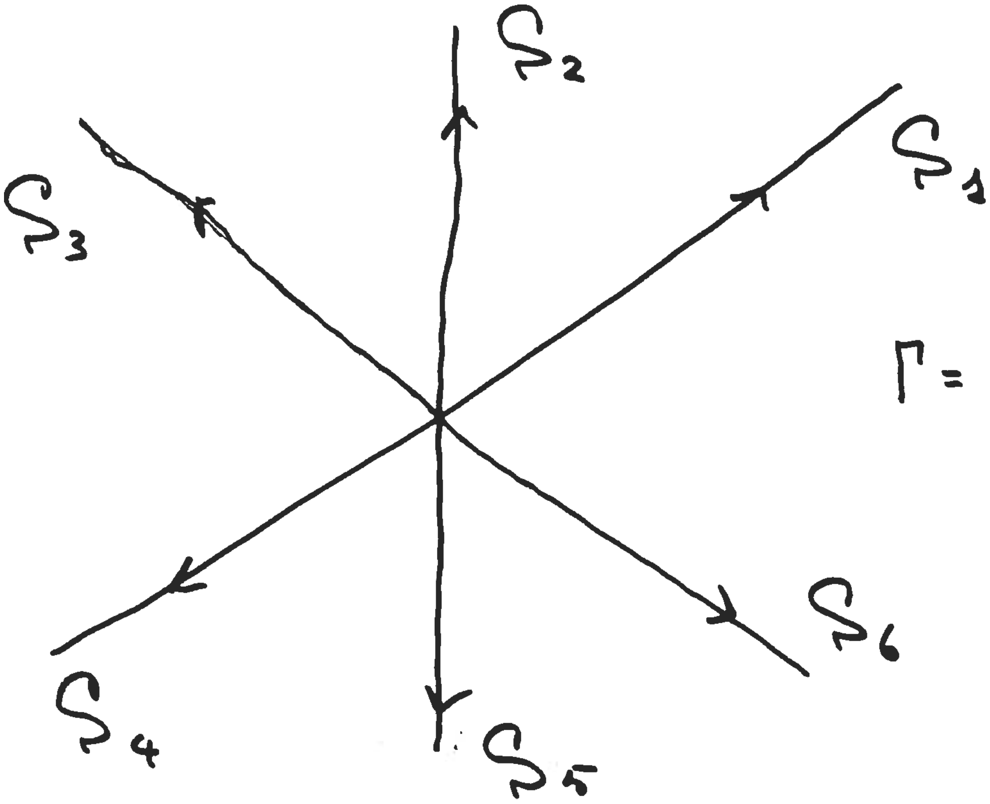
S – collection of Stokes matrices

Γ – collection of anti-Stokes rays

$$\boxed{u(x) = \text{"}\operatorname*{res}_{\lambda=\infty}\text{"}\, Y(\lambda, x)}$$

Example. $P\mathrm{II}$, $\alpha = 0$.

Γ:



$$\Gamma = \bigcup_{k=1}^{6} \Gamma_k$$

$$S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, \dots$$

$$s_4 = -s_1, \quad s_5 = -s_2, \quad s_6 = -s_3$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0 \iff S_1 \cdots S_6 = I$$

$$G(\lambda) = e^{i\theta(\lambda)} S_k e^{-i\theta(\lambda)} \qquad \lambda \in \Gamma_k$$

$$\theta(\lambda) = \theta(\lambda, x) = -\left(\frac{4}{3}\lambda^3 + x\lambda\right)\sigma_3$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{Y(\lambda) \equiv Y(\lambda, x):}$$

- $Y(\lambda) \in H(\mathbb{C} \setminus \Gamma)$

- $Y_+(\lambda) = Y_-(\lambda)\, G(\lambda), \quad \lambda \in \Gamma$

- $Y(\lambda) \longmapsto I, \quad \lambda \to \infty$

$$\boxed{u(x) = 2\,(m_1)_{12}}$$

$$Y(\lambda) \sim I + \frac{m_1}{\lambda} + \cdots \qquad \lambda \to \infty$$

$$\boxed{u_{xx} = xu + 2u^3}$$

Linear system:

$$\Psi(\lambda) := Y(\lambda)\, e^{i\theta(\lambda)}$$

$$\frac{d\Psi}{d\lambda} = A(\lambda)\,\Psi$$

Note:

$i\frac{d\theta}{d\lambda} \equiv$ BNF of $A(\lambda)$

also:

$\sigma_2 A(-\lambda)\sigma_2 = -A(\lambda)$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$A(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$$

$$= \begin{pmatrix} -4i\lambda^2 - ix - 2iu^2 & i\lambda u - 2u_x \\ -i\lambda u - 2u_x & 4i\lambda^2 + ix + 2iu^2 \end{pmatrix}$$

The List of Painlevé RH problems

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi \quad , \quad \theta(\lambda, x)$$

$$\left(\frac{d\theta}{d\lambda} - \text{BNF of } A(\lambda) \right)$$

PI:

$$A(\lambda) = \sum_{k=-1}^{4} \lambda^k A_k \qquad \sigma_1 A(-\lambda)\sigma_1 = -A(\lambda)$$

$$\theta(\lambda) = (\lambda^5 + x\lambda)\sigma_3$$

PII:

$$A(\lambda) = \sum_{k=-1}^{2} \lambda^k A_k \qquad \sigma_2 A(-\lambda)\sigma_2 = -A(\lambda)$$

$$\theta(\lambda) = (\lambda^3 + x\lambda)\sigma_3$$

or

$$A(\lambda) = \sum_{k=0}^{2} \lambda^k A_k \qquad \theta(\lambda) = (\lambda^3 + x\lambda + \nu \ln\lambda)\sigma_3$$

$P\overline{\text{III}}$:

$$A(\lambda) = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2}$$

$$\theta(\lambda) = x\left(\lambda + \frac{1}{\lambda}\right)\sigma_3$$

$P\overline{\text{IV}}$:

$$A(\lambda) = \sum_{k=0}^{3} A_k \lambda^k \qquad \sigma_3 A(-\lambda)\sigma_3 = -A(\lambda)$$

$$\theta(\lambda) = \left(\lambda^4 + x\lambda^2\right)\sigma_3$$

or

$$A(\lambda) = A_1\lambda + A_0 + \frac{A_{-1}}{\lambda} \qquad \theta(\lambda) = \left(\lambda^2 + x\lambda\right)\sigma_3$$

$P\overline{\text{V}}$:

$$A(\lambda) = A_0 + \frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} \qquad \theta(\lambda) = (x\lambda)\sigma_3$$

$P\overline{\text{VI}}$:

$$A(\lambda) = \frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - x}$$

Garnier ; Jimbo, Miwa; Flaschka, Newell

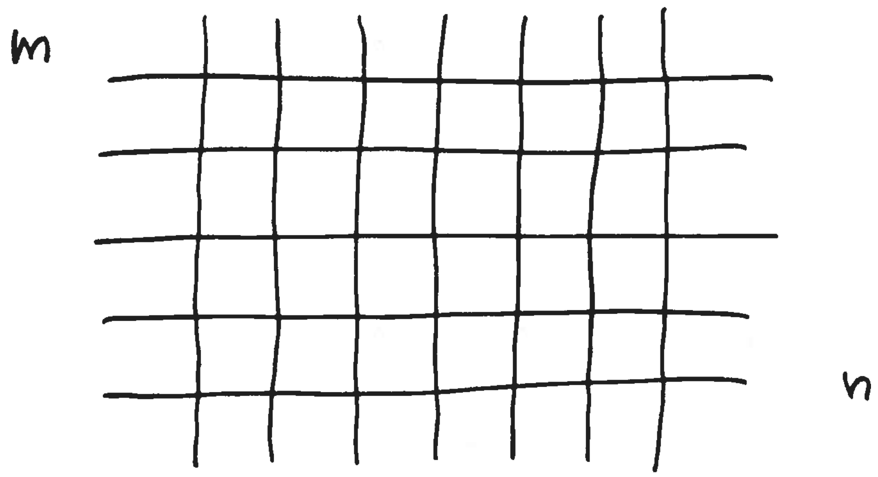
<u>The principal point</u>:

$$\boxed{\text{The RH representation} \equiv \text{non-abelian analog of contour integral representation.}}$$

$$\Downarrow \; !$$

The possibility of global asymptotic analysis of the Painlevé functions.

Ising Model



Energy Functional:

$$E(\sigma) = -E^{v} \sum_{n,m=0}^{L} \sigma_{mn}\,\sigma_{m+1\,n} - E^{h} \sum_{n,m=0}^{L} \sigma_{mn}\,\sigma_{mn+1}$$

$\sigma_{mn} = \pm 1$ - value of spin variable at (m,n) - side

Partition function:

$$Z_L = \sum_{\sigma} e^{-\beta E(\sigma)} \quad , \quad \beta = \frac{1}{kT}$$

free energy:

$$f = -kT \lim_{L\to\infty} \frac{1}{L^2} \ln Z_L$$

two-point correlation function:

$$\langle \sigma_{00}\, \sigma_{MN} \rangle_L = \frac{1}{Z_L} \sum_{\sigma}{}' \sigma_{00}\, \sigma_{MN}\, e^{-\beta E(\sigma)}$$

$$\boxed{\langle \sigma_{00}\, \sigma_{MN} \rangle = \lim_{L\to\infty} \langle \sigma_{00}\, \sigma_{MN} \rangle_L}$$

$$M_0 = \lim_{n\to\infty} \langle \sigma_{00}\, \sigma_{nn} \rangle^{1/2}$$

(1944) Onsager: Free energy, T_c:

$$\sinh \frac{2E^v}{kT_c} \sinh \frac{2E^h}{kT_c} = 1.$$

(1936. Peierls, 1941. Kramers & Wannier)

$T < T_c$:

$$\langle \sigma_{00} \sigma_{nn} \rangle \sim (1 - a^{-2})^{1/4}$$

Kaufman & Onsager 1948

$$a = \sinh \frac{2E^v}{kT} \sinh \frac{2E^h}{kT}$$

$T > T_c$:

$$\langle \sigma_{00} \sigma_{nn} \rangle \sim \frac{\sqrt{\pi}}{n^{1/2}} \frac{a^n}{(1 - a^2)^{1/4}}$$

Wu 1966 ($\langle \sigma_{00} \sigma_{0n} \rangle$)

$T = T_c$:

$$\langle \sigma_{00}\sigma_{nn}\rangle \sim e^{1/4} A^{-3} 2^{1/12} n^{-1/4}$$

$$A = e^{1/12 - \zeta'(-1)}$$

Wu 1966

$$\langle \sigma_{00}\sigma_{mn}\rangle \sim e^{1/4} 2^{5/24} A^{-3} k^{-1/4}$$

$$k = \sqrt{\alpha m^2 + \beta n^2} \qquad \alpha = \frac{2\sinh\frac{2E^h}{kT_c}}{\sinh\frac{2E^h}{kT_c} + \sinh\frac{2E^v}{kT_c}}$$

Cheng & Wu 1967 $\qquad \beta = (E^h \to E^v)$

H. Pinson 2011, D. Chelkak, C. Hongler, K. Izyurov 2012

$(E^v = E^h)$

Scaling Theory.

Wu, McCoy, Tracy, Barouch.

$$G_{\pm}(\tau) := \lim_{\substack{k \to \infty \\ t \to \mp 0}} |1 - e^{-2t}|^{-1/4} \langle \sigma_{00} \sigma_{mn} \rangle$$

$e^{t} := a$ $\qquad$ $k|t| \equiv \tau$, fixed.

$\left(k|T - T_c| = O(1) \right)$

$$G_{\pm}(\tau) = \frac{1 \mp \eta(\tau/2)}{2\eta^{1/2}(\tau/2)} \exp \int_{\tau/2}^{\infty} \frac{1}{4} \frac{(1-\eta^2)^2 - \eta'^2}{\eta^2} x \, dx$$

$\eta \equiv \eta(x)$:

$$\boxed{\eta'' = \frac{1}{\eta}(\eta')^2 - \frac{1}{x}\eta' + \eta^3 - 1/\eta \quad - \quad P\,\overline{\underline{III}}.}$$

$T > T_c$ asymptotic $\Rightarrow$ B.C.

$$\eta(x) \sim 1 - \frac{1}{\sqrt{\pi}} x^{-1/2} e^{-2x} \qquad (1)$$

$x \to +\infty$

Key question:

$(1) \Rightarrow$? $\quad C_+(\tau) \sim \text{const}\ \tau^{-1/4}$!

$\tau \to 0$

$\uparrow$

matching with $T = T_c$ asymptotics

McCoy, Tracy, Wu (1977)

Tracy (1991) $\mapsto$ the <u>constant</u>.

McCoy - Tracy - Wu connection formulae

$$\eta \sim 1 - \gamma \sqrt{\pi}\, x^{-1/2} e^{-2x}, \quad x \to +\infty$$

$$|\gamma| \leq \frac{1}{\pi}$$

$$\Downarrow \; !$$

$$\eta \sim B x^{\sigma}, \quad x \to 0, \quad |\gamma| < \frac{1}{\pi}$$

$$\boxed{\sigma = \frac{2}{\pi} \arcsin \pi\gamma, \quad B = 2^{-3\sigma} \frac{\Gamma\left(\frac{1-\sigma}{2}\right)}{\Gamma\left(\frac{1+\sigma}{2}\right)}}$$

$$\eta \sim -x \left(\ln \frac{x}{4} + \gamma_E \right), \quad x \to 0, \quad \gamma = \frac{1}{\pi}$$

Alternative representation.

$$G_-(\tau) = \exp\left[-\int_{2\tau}^{\infty} \frac{\sigma(x)}{x}\,dx\right]$$

$\sigma(x)$:

$$(x\sigma'')^2 = (\sigma - x\sigma' + 2\sigma'^2)^2 - 4\sigma'^2(\sigma'^2 - 1/4) \quad - P\overline{\underline{V}}$$

$$\sigma(x) = \begin{cases} -1/4 + O(x\ln x) & x \to 0 \\ \frac{1}{2\pi} x^{-1} e^{-x}(1 + O(1/x)) & x \to \infty \end{cases}$$

Jimbo & Miwa 1980

(McCoy & Perk 1987 - equivalence to $P\overline{\underline{III}}$ - formula.)

Claeys, Krasovsky, I 2011:

$$\ln \langle \sigma_{00}\sigma_{nn} \rangle = \frac{1}{4}\log(1-e^{-2t}) - \frac{1}{4}\log 2nt$$

$$+ \int_0^{2nt} \frac{\sigma(x)+1/4}{x}\,dx + \log G\left(\tfrac{1}{2}\right)G\left(\tfrac{3}{2}\right)$$

$$+\, o(1) \quad , \; n \to \infty$$

$\uparrow$ uniformly in $0 \le t < t_0$

(+ generalization)

Painlevé before large N limit.

$$N < \infty, \qquad t := \left(\sinh 2E^v/kT \, \sinh 2E^h/kT \right)^{-2}$$

$$\zeta(t) := t(t-1) \frac{d}{dt} \ln \langle \sigma_{00} \sigma_{NN} \rangle - \frac{1}{4} t$$

$$\boxed{\begin{aligned} \left(t(t-1) \frac{d^2\zeta}{dt^2} \right)^2 &= N^2 \left[(t-1) \frac{d\zeta}{dt} - \zeta \right]^2 \\ &\quad - 4 \frac{d\zeta}{dt} \left[(t-1) \frac{d\zeta}{dt} - \zeta - \frac{1}{4} \right] \left(t \frac{d\zeta}{dt} - \zeta \right) \end{aligned}}$$

P$\underline{\text{VI}}$ (σ-form) Jimbo, Miwa (1981)

⊕ Discrete P$\underline{\text{VI}}$ ($\zeta_N \to \zeta_{N+1}$),
Toda for $\zeta_N(t)$

More on Painlevé and Stat. Mechanics

XY spin-½ model, XXO, XXX, XXZ

six-vertex model

see McCoy's review and

Palmer, Perk,

Forrester, Witte,

Izergin, Korepin, Slavnov, I,

Kanzieper

Bleher, Fokin

Self-similar reduction of integrable PDEs.

$V_t + V_{zzz} - 6V^2 V_z = 0$ - mKdV equation

put

$$V(t,z) = \frac{1}{(3t)^{1/3}}\, u\left(\frac{z}{(3t)^{1/3}}\right)$$

then $u \equiv u(x)$ satisfies PII:

$$\boxed{u_{xx} = xu + 2u^3 + \text{const}}$$

Ablowitz, Segur (1977)

(Zakharov, Manakov, Shabat)

Moreover, matching with the asymptotics of $V(t,z)$ as $t \to \pm\infty$, $z/t = O(1)$, the following 1-parameter family of PII appear: (Ablowitz - Segur)

$$u(x) \equiv u(x,a) \sim \frac{a}{2\sqrt{\pi}}\, x^{-1/4} e^{-\frac{2}{3}x^{3/2}}$$

$|a|<1$

$x \to +\infty$

! $$u(x,a) \sim (-x)^{-1/4}\, d \cos\left(\frac{2}{3}(-x)^{3/2} - \frac{3}{4} d^2 \ln(-x) + \varphi\right)$$

$x \to -\infty$

$$\boxed{\begin{gathered} d^2 = -\frac{1}{\pi}\ln(1-a^2), \quad d>0 \\ \varphi = -\frac{3}{2} d^2 \ln 2 - \frac{3\pi}{4} - \arg \Gamma\left(-i\frac{d^2}{2}\right) + \pi n \\ \operatorname{sign} a = (-1)^n \end{gathered}}$$

Ablowitz - Segur (1977) - proof of the formula for d

formula for φ was proven by:

Suleimanov; Clarkson, McLeod; Deift, Zhou

This type of Painlevé appearance:

QFT – PIII as reduction of sine-Gordon, 2D-Toda models

(Cecotti, Vafa, Fateev, Lukyanov...)

Differential geometry of surfaces – PIII
(Bobenko, et al)

Quantum cohomology, Frobenius manifolds – PIII, PVI
(Dubrovin, et al ; Dorfmeister, Guest, Rossman)

Josephson junctions – PIII reduction of 2D elliptic sine-Gordon (Novokshenov, Shagalov)

The polyelectrolytes theory (Tracy, Widom)

Hermitian Matrix Model

$$\Omega_N = \{ M = N \times N \text{ Hermitian matrix} \}$$

$$d\mu_N = \frac{1}{Z_N} e^{-N \operatorname{Tr} V(M)} DM$$

$$DM = \prod_j dM_{jj} \prod_{j<k} dM_{jk}^R \, dM_{jk}^I$$

$$Z_N = \int_{\Omega_N} e^{-N \operatorname{Tr} V(M)} DM \quad \text{- partition function.}$$

$$V(\lambda) = \sum_{j=1}^{2m} t_j \lambda^j \qquad t_{2m} > 0$$

Principal questions.

$$Z_N \longmapsto \; ? \quad , \; N \to \infty$$

eigenvalue statistics $\longmapsto$? , $N \to \infty$

Universality.

D. Bessis, C. Itzykson, J.B. Zuber

F. Dyson, M. Mehta

Quartic potential

$$V(\lambda) = \frac{1}{4t^2}\lambda^4 + \left(1-\frac{2}{t}\right)\lambda^2 \qquad , \quad t>0$$

(1.) $Z_N \longmapsto F_N = -\frac{1}{N^2}\ln Z_N$ - free energy

$$F_N(t) = F_N^{\text{Gauss}} + \int_t^{\infty} \frac{t-s}{s^2}\left[R_N(R_{N+1}+R_{N-1}) - \frac{1}{2}\right] ds$$

$R_n = R_n(t)$:

$$\frac{n}{N} = R_n\left(1-\frac{2}{t}\right) + \frac{1}{4t^2} R_n (R_{n-1} + R_n + R_{n+1})$$

$R_n = 0 \qquad n \leq 0 \qquad\qquad$ BIZ

(d-PI $\equiv$ discrete string equ. = Freud equ.)

Also:

$$\{P_n(\lambda)\}_{n=1}^{\infty} \quad : \quad \int_{-\infty}^{\infty} P_n(\lambda) P_k(\lambda) e^{-NV(\lambda)} d\lambda = h_n \delta_{nk}$$

$$P_n(\lambda) = \lambda^n + \dots$$

$$\lambda P_n = P_{n+1} + R_n P_{n-1}$$

$$R_n = \frac{h_n}{h_{n-1}}$$

The phase transition: $t_c = 1$, i.e.

$t > 1$ – "one cut"

$t < 1$ – "two cut"

$t > 1$:

$$R_n \sim \frac{t}{3}\left(2 - t + \sqrt{(t-2)^2 + 3\frac{n}{N}}\right)$$

$$\left(+ \sum_{k=1}^{\infty} N^{-2k} f_k\left(\frac{n}{N}; t\right)\right)$$

$t < 1$:

$$R_n \sim t\left(2 - t + (-1)^{n+1}\sqrt{(t-2)^2 - \frac{n}{N}}\right)$$

$N \to \infty$, $n = N, N \pm 1$

Bleher, I. (1998)

Scaling Theory.

$$N \to \infty, \quad t \to 1, \quad N^{2/3}(t-1) = O(1)$$

$$\boxed{R_n(x) = 1 - N^{-1/3}\, 2^{2/3}\, (-1)^n\, u_{HM}(x) + 2^{-5/3}\, N^{-2/3}\, v_{HM}(x) + O(1/N)}$$

$$N \to \infty, \quad t = 1 + 2^{-2/3} N^{-2/3} x$$

$\boxed{u_{HM}:}$ $\quad u_{xx} = xu + 2u^3 \quad - \quad P\,\underline{\mathrm{II}}.$

$$u(x) \simeq \frac{1}{2\sqrt{\pi}}\, x^{-1/4}\, e^{-\frac{2}{3}x^{3/2}}$$

$$x \to +\infty$$

! Important: $\quad u(x) \simeq \sqrt{-x/2}, \quad x \to -\infty$

$$(v' = x + 2u^2)$$

Douglas, Seiberg, Shenker

Genković, Moore

Periwal, Shevitz

Bleher, I (2003)

! Also:

$$Z_N(t)/Z_N^{Gauss} \sim F_{TW}\left((t-1)\,2^{2/3} N^{2/3}\right) \times Z_N^{reg}(t)$$

$$F_{TW}(x) = \exp\left\{ \int_x^\infty (x-y)\, u_{HM}^2(y)\, dy \right\}$$

Bleher, I (2005)

2. 2D Quantum Gravity

$$Z_N(t) = \int_{\{M\}} e^{-N\,\mathrm{Tr}\left(\frac{1}{2}M^2 + t M^4\right)}\, dM \qquad t > 0$$

$$\mathring{Z}_N(g) = \int_{\{M\}} e^{-N\,\mathrm{Tr}\left(\frac{g}{2}M^2 + \frac{1}{4} M^4\right)}\, dM$$

$$\boxed{Z_N(t) = 2^{-N^2/2}\, t^{-N^2/4}\, \mathring{Z}_N\left(\frac{1}{2\sqrt{t}}\right)}$$

t

- $$\log Z_N(t)/Z_N(0) \cong \sum_{g=0}^{\infty} N^{2-2g} E_g(t)$$

$$E_g(t) = \sum_{n\geq 1}{}' (-t)^n \frac{1}{n!} \mathcal{X}_g(n)$$

analytic : $|t| < \frac{1}{48}$

<u>proven</u> for $t > 0$ (Ercolani & McLaughlin)

$\mathcal{X}_g(n) = \#\{$ g-maps with n 4-valent vertices $\}$

"=" # { tilings of g- surface with n squares }

Bessis, Itzykson, Zuber (1980)

also proven for

$$-\frac{1}{48} < t < 0 \quad \text{(Duits \& Kuijlaars)}$$

$t = -\frac{1}{48}$ - singularity of $E_g(t)$.

Double scaling near $t = -\frac{1}{48}$.

$$t = -\frac{1}{48} - 2^{-18/5} N^{-4/5} x$$

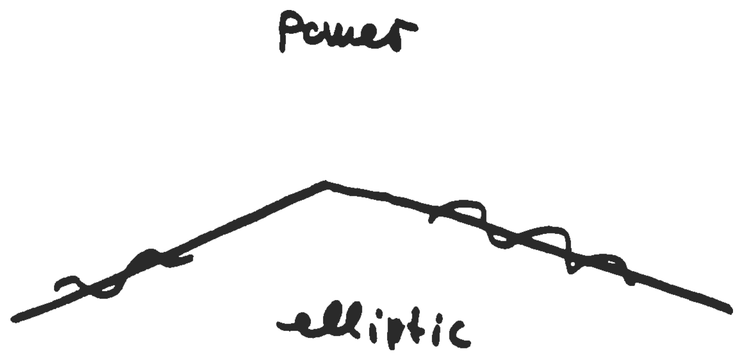
$$R_N \sim 2 - c N^{-2/5} u(x), \quad c = 2^{3/5} 3^{2/5}$$

$$u_{xx} = 6u^2 + x \quad - \text{PI}$$

$$u(x) \sim \sqrt{-\frac{x}{6}} + \sum_{p=1}^{\infty} (-x)^{\frac{1}{2} - \frac{5}{2}p} c_p, \quad x \to \infty$$

$$-\frac{\pi}{5} < \arg x < \frac{7\pi}{5}$$

Brézin & Kazakov; Douglas & Shenker; Gross & Migdal; David; Kitaev; Fokas, Kitaev, I; Duits & Kuijlaars



$$u(x) = e^{\frac{i\pi}{5}} \sqrt{\frac{|x|}{6}}$$

$$+ \frac{1}{\sqrt{8\pi}} e^{-i\pi/20} \left(\frac{2}{3}\right)^{1/8} i \, |x|^{-1/8}$$

$$\times \exp\left\{ -\frac{8i}{5} \left(\frac{3}{2}\right)^{1/4} |x|^{5/4} \right\}$$

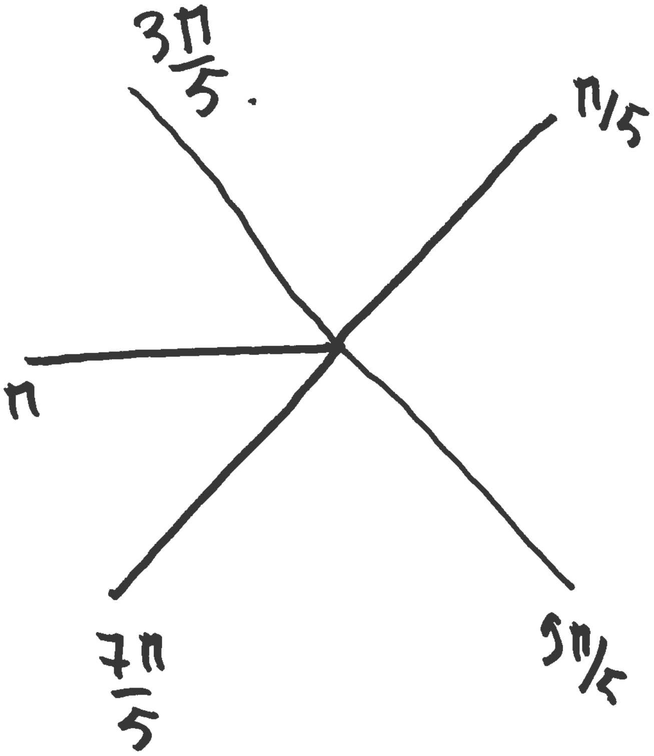
$$(1 + o(1))$$

$$x \to \infty, \quad \arg x = \frac{7\pi}{5}$$

A. Kapaev.

(triply truncated solution)

PI - sectors:



$$u_{xx} = 6u^2 + x$$

$$\Downarrow$$

$$\frac{1}{N^2} \log Z_N(t) \underset{N\to\infty\ ,\ t = -\frac{1}{48} - 2^{-19/5} N^{-4/5} x}{\longmapsto} -F(x)$$

$$F''(x) = -u(x)$$

Matching with the non-critical expansion.

$$\varkappa_g(n) \sim \frac{48^n \, n!}{n^{1+\frac{5}{2}(1-g)}} c_g^0 \iff E_g(t) \sim a_g \left(t + \frac{1}{48}\right)^{\frac{5}{2}(1-g)}$$

(Di Francesco , Guo)

$$\Downarrow$$

$$\sum_{g=0}^{\infty}{}' N^{2-2g} E_g(t) \text{ "}\sim\text{" } \sum_{g=0}^{\infty} N^{2-2g} a_g \left(t + \frac{1}{48}\right)^{\frac{5}{2}(1-g)}$$

$$\text{"}=\text{"} \sum_{g=0}^{\infty}{}' c_g (-x)^{\frac{5}{2} - \frac{5}{2} g}$$

$$\mathcal{L}_g(n) \sim \frac{48^n \, n!}{n^{1+5/2(1-g)}} \, c_g^0$$

$$u(x) \cong \sqrt{-\frac{x}{6}} + \sum_{g=1}^{\infty} (-x)^{\frac{1}{2}-\frac{5}{2}g} c_g$$

$$\boxed{c_g^0 = \frac{4}{15} \left(2\cdot 3^5\right)^{\frac{1}{2}(g-1)} \frac{1}{\Gamma\left(\frac{5}{2}(g-1)-1\right)} \, c_g}$$

$$\left(c_g \sim S \left(8\sqrt{3}/5\right)^{-2g+1/2} \Gamma\left(2g-\tfrac{1}{2}\right) \right)$$

(Garoufalidis, Kashaev, Mariño, I)

$\overline{M}_{g,n}$ ≡ moduli space of 

$$\psi_i = c_1(\mathcal{L}_i) \quad i = 1, \dots, n$$

$\mathcal{L}_i$ – vector bundle over $\overline{M}_{g,n}$

$$\pi^{-1}(\Sigma'_g) = T^*_{a_i}\Sigma'_g$$

$$\langle \tau_2^{3g-3} \rangle_g := \int_{\overline{M}_{g,3g-3}} \psi_1^2 \wedge \dots \wedge \psi_{3g-3}^2$$

$$\boxed{\langle \tau_2^{3g-3} \rangle_g = K_g c_g}$$

$$K_g \text{ "=" } - \frac{4^g (3g-3)!}{(5g-5)(5g-3)}$$

$$\Downarrow$$

$$\int (x-y)\, u(y)\, dy$$

= non-perturbative definition

of the partition function

of 2D quantum gravity.

More on this type of the Painlevé equations appearance.

1. Other graph counting problems. Intersection theory of the moduli spaces of curves. Witten - Kontsevich theorem.

   Di Francesco, Ginsparg, J. Zinn-Justin
   Itzykson, Zuber, Eguchi, Yamada, S-K Yang
   Okounkov, Pandharipande,
   Adler, Van Moerbeke,
   Gao, Garoufalidis, Marino, Brèzen, Hikami

2. Hele - Shaw flow, normal matrix model, conformal maps & integrable hierarchies

   Zabrodin, Wiegmann, Mineev-Weinstein and others.

Painlevé before large N limit.

$$R_n(t,N) = \frac{t}{\sqrt{N}}\, u(x)\,, \quad x = (t-2)N^{1/2}$$

$u(x) \equiv u(x,n)$:

$$\boxed{\frac{d^2u}{dx^2} = \frac{1}{2u}\left(\frac{du}{dx}\right)^2 + \frac{3}{2}u^3 + 4xu^2 + 2\left(x^2+\frac{n}{2}\right)u - \frac{n^2}{2u}}$$

$\overline{\text{PIV}}$ Kitaev (1991)

More on this:

Magnus, Chen, Ismail, Fokas, Kitaev, I

- Painlevé for semiclassical orthogonal polynomials

Adler, Van Moerbeke - matrix integrals and Painlevé

Forrester, Frankel, Witte - τ-function theory approach

Kanzieper - replica

3. Eigenvalue statistics.

Bulk :

$$E_0^{(N)}\left(\lambda_0 - \frac{cx}{N\rho(\lambda_0)},\ \lambda_0 + \frac{cx}{N\rho(\lambda_0)}\right)$$

$$\underset{N\to\infty}{\longmapsto} \quad E_0(x)$$

$$E_0(x) = \det\left(1 - K_{\sin}\big|_{(0,2x)}\right)$$

$$= \exp\left(\int_0^x \frac{\sigma(t)}{t}\,dt\right)$$

$$\boxed{\begin{gathered}(x\sigma'')^2 + 4\left(4\sigma - x\sigma' - (\sigma')^2\right)(\sigma - x\sigma') = 0 \\ -\mathrm{P}\overline{\underline{\mathrm{V}}} \\ \sigma(x) \sim -\frac{2}{\pi}x \quad , x \to 0. \qquad \text{JMMS}\end{gathered}}$$

$$E_0(x) \sim x^{-1/4} e^{-\frac{1}{2}x^2 + c_0}, \quad x \to +\infty$$

$$c_0 = 3\zeta'(-1) + \frac{1}{12}\ln 2$$

Dyson, Widom, Suleimanov, Krasovsky, Ehrhardt

Edge:

$$\rho(\lambda) \sim c(\lambda - b)^{\frac{4k+1}{2}}$$

b

$$\mathrm{Prob}\left(c N^{\frac{2}{4k+3}} (\lambda_{max}^{(N)} - b) < x \right)$$

$$\xrightarrow[N\to\infty]{} F_{TW}^{(k)}(x)$$

$$F_{TW}^{(0)}(x) \equiv F_{TW}(x) \quad (\text{Tracy-Widom})$$

$F_{TW}^{(k)}(x)$ = in terms of $P\,\mathrm{II}^{(k)}$

Claeys, Krasovsky, I

More on this type of Painlevé appearance

Tracy, Widom

Adler, Van Moerbeke

Forrester, Witte

Osipov, Kanzieper

Summary on Painlevé in R M.

- Distributions of classical matrix models are given in terms of Painlevé functions

- Universal properties of the distribution functions of RMT are described by Painlevé transcendents

1990: Brézin & Kazakov; Douglas & Shenker;
Gross & Migdal

1992: Tracy & Widom; Mahoux & Mehta

- Bowick, Crnković, Di Francesco, Forrester, Moore, Periwal, Seiberg, Shevitz
- Adler, Shiota, van Moerbeke
- Fokas, Kitaev, I

60-70: Dyson, Mehta, Gaudin,
des Cloizeaux, Lenard

70-80: Barouch, McCoy, Tracy, Wu
Jimbo, Miwa, Mori, Sato

↑

↓

≈ 2000:

- Bleher, I; Baik, Deift, Johansson;
Deift, Kricherbauer, McLaughlin, Venakides, Zhou;
Okounkov; Borodin, Okounkov

- Baik, Forrester, Rains, Witte;
Tracy, Widom; Palmer; Harnad, I;
Borodin, Deift;
Adler, Harnad, Orlov, Shiota, van Moerbeke

- $N < \infty \longrightarrow$ Painlevé "<u>functions</u>"

  $RMT \rightarrow P$

- $N = \infty \longrightarrow$ Painlevé <u>transcendents</u>

  $RMT \leftarrow P$

An explanation of the appearance of Painlevé equations

1. A Riemann-Hilbert representation of the original physical problem

2. Direct identification of the original RH problem with the RH problem from the Painlevé list - <u>Painlevé equations before or after the large N limit.</u>

3. Appearance of the Painlevé RH problem as a parametrix of the asymptotic solution of the original RH problem - <u>Double scaling limits, transition asymptotic regimes</u>

≡ appearance of the Airy, Bessel, etc. in the asymptotic analysis of oscillatory integrals.

Important General Idea:

LSD:

$$y = \int_\Gamma e^{t d(\lambda)} g(\lambda)\, d\lambda$$

$$e^{t d(\lambda)} g(\lambda) \underset{t \to \infty}{\sim} e^{t d(\lambda_0)} \underbrace{g(\lambda_0)}_{\text{const}}$$

$$\Downarrow$$

$y \sim$ Classical SF

NLSD:

$$e^{tD(\lambda)} G(\lambda) e^{-tD(\lambda)} \sim e^{tD_0(\lambda)} \underbrace{G(\lambda_0)}_{\text{const}} e^{-tD_0(\lambda)}$$

$$\Downarrow$$

$Y \sim$ Nonlinear SF $\equiv$ Painlevé

($\oplus$ Theta !)

1.1 The RH representation of the correlation functions.

- Determinant representation of the correlation functions:

$$\boxed{\det(1-K)} \qquad K : L_2(\Gamma) \circlearrowleft$$

$$K(\lambda,\lambda') = \frac{f^T(\lambda)\, h(\lambda')}{\lambda-\lambda'}$$

$$f(\lambda) = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}, \quad h(\lambda) = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix}$$

(integrable integral operators)

Examples

Sine-Kernel, Airy-Kernel, Bessels-kernel,

Hermite-Kernel, Darboux-Kernel

Painlevé-kernels

Gaudin, Lenard,

Jimbo, Miwa, Mori, Sato

Tracy, Widom

Forrester

Izergin, Korepin, Slavnov, I

Akemann; Bowick, Brézin, Marinari, Parisi,

Bleher, I

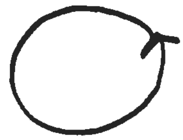
Claeys, Kuijlaars, Vanlessen

Claeys, Krasovsky, I

Kuijlaars, Östensson, I

! Toeplitz determinants

$$\varphi_j = \int_{|\lambda|=1} \varphi(\lambda)\, \lambda^{-j-1} \frac{d\lambda}{2\pi i}$$



$|\lambda| = 1$

$$T_n[\varphi] = \{ \varphi_{j-k} \}_{j,k = 0, \dots n-1}$$

$$\det T_n[\varphi] = \det(1 - K_n)$$

$$K_n : L_2(\Gamma) \circlearrowleft \qquad \Gamma : |\lambda| = 1$$

$$K_n(\lambda, \lambda') = \frac{(\lambda/\lambda')^n - 1}{\lambda - \lambda'} \; \frac{1 - \varphi(\lambda')}{2\pi i}$$

(Deift)

$\Rightarrow$ correlation functions of Ising and quantum spin model, random permutations, random tiling

- The RH representation:

$$R = (1-K)^{-1}K\,, \qquad R(\lambda,\lambda') = \frac{F^T(\lambda)\,H(\lambda')}{\lambda-\lambda'}$$

$$F(\lambda) = Y_+(\lambda)\,f(\lambda)\,, \qquad H(\lambda) = \left(Y_+^T(\lambda)\right)^{-1} h(\lambda)$$

- $Y(\lambda) \in H\left(\mathbb{C}\setminus\Gamma\right)$

- $Y_-(\lambda) = Y_+(\lambda)\,G(\lambda)\,, \quad \lambda\in\Gamma$

- $Y(\lambda) \sim I\,, \quad \lambda\to\infty$

$$G(\lambda) = I_m + 2\pi i\, f(\lambda)\,h^T(\lambda)$$

$$\left(G_{jk} = \delta_{jk} + 2\pi i\, f_j(\lambda)\,h_k(\lambda)\right)$$

(Izergin, Korepin, Slavnov, I.)

Remark. An alternative RH problem for Toeplitz determinants.

$$G_T(\lambda) = \begin{pmatrix} 2-\varphi & \lambda^n(\varphi-1) \\ \lambda^{-n}(1-\varphi) & \varphi \end{pmatrix}$$

$$\equiv \begin{pmatrix} \lambda^n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\lambda^{-n}\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-n} & 0 \\ -1 & \lambda^n \end{pmatrix}$$

$$\tilde{Y}(\lambda) := \begin{cases} Y(\lambda) \begin{pmatrix} \lambda^n & -1 \\ 1 & 0 \end{pmatrix} & |\lambda| < 1 \\ \\ Y(\lambda) \begin{pmatrix} \lambda^n & 0 \\ 1 & \lambda^{-n} \end{pmatrix} & |\lambda| > 1 \end{cases}$$

- $\tilde{Y}(\lambda) \in H(\mathbb{C} \setminus \Gamma)$ $\qquad \Gamma : |\lambda| = 1$

- $\tilde{Y}_+(\lambda) = \tilde{Y}_-(\lambda) \begin{pmatrix} 1 & \lambda^{-n}\varphi(\lambda) \\ 0 & 1 \end{pmatrix}$ $\quad \lambda \in \Gamma$

- $\tilde{Y}(\lambda) = \left(I + O\left(1/\lambda\right)\right) \lambda^{n\sigma_3}$, $\lambda \to \infty$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Important:

$\tilde{Y}_{11}(\lambda) = P_n(\lambda)$ - OPUC:

$$\int_\Gamma P_n(\lambda) \overline{P_j(\lambda)} \varphi(\lambda) \frac{d\lambda}{i\lambda} = h_n \delta_{nj}, \quad P_n(\lambda) = \lambda^n + \dots$$

also: $\quad \det T_{n+1} / \det T_n \equiv \frac{1}{2\pi} h_n = \tilde{Y}_{12}(0)$

Baik, Deift, Johansson (1999)

1.2 The RH representation of orthogonal polynomials ($\Rightarrow$ Hermitian matrix model)

$$\Gamma = \mathbb{R}, \qquad G(\lambda) = \begin{pmatrix} 1 & w(\lambda) \\ 0 & 1 \end{pmatrix}$$

$$w(\lambda) = e^{-V(\lambda)}, \quad V(\lambda) = \sum_{j=1}^{2m} t_j \lambda^j, \quad t_{2m} > 0$$

- $Y(\lambda) \in H(\mathbb{C} \setminus \mathbb{R})$

- $Y_+(\lambda) = Y_-(\lambda)\, G(\lambda)$

- $Y(\lambda) = (I + O(1/\lambda))\, \lambda^{n\sigma_3}, \quad \lambda \to \infty$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y_{11}(\lambda) = P_n(\lambda) \text{ - OPRL:}$$

$$\int_{-\infty}^{\infty} P_n(\lambda) P_e(\lambda)\, \omega(\lambda)\, d\lambda = h_n \delta_{ne} \quad , \; P_n(\lambda) = \lambda^n + \dots$$

also:

$$D_{n+1} / D_n \equiv h_n = \frac{i}{2\pi} \lim_{\lambda \to \infty} \lambda^{n+1} Y_{12}(\lambda)$$

note:

$$Z_N = \text{const} \int \dots \int \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} \omega(\lambda_j)\, d\lambda_1 \dots d\lambda_N$$

$$= \text{const}\; N!\; D_N$$

$$D_n = \det \left\{ \int_{-\infty}^{\infty} \lambda^{k+j} \omega(\lambda)\, d\lambda \right\}_{k,j = 0 \dots n-1}$$

– Hankel determinant.

Fokas, Its, Kitaev, I (1991)

No explanation !

(Original problem – not integrable)

- Three-dimensional wave collapse and P_{II}.

  (Zakharov, Kuznetsov, Musher; Novokshenov)

- Soshnikov's result

- Current fluctuations in ASEP

  (Tracy, Widom)

$\Downarrow$

The Painlevé transcendents are <u>indeed</u> the <u>nonlinear special functions</u>.

Appendix 1.

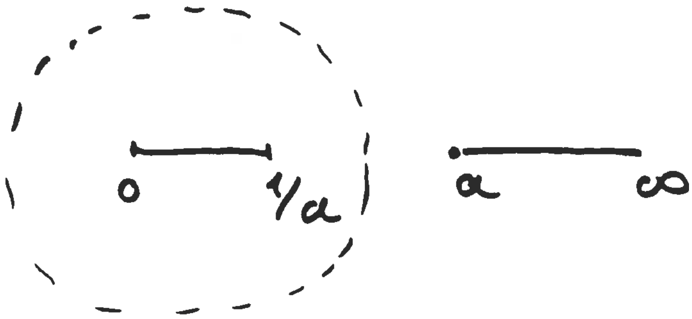
Ising model. Painlevé VI.

(Jimbo - Miwa '81)

$$\langle \sigma_{00}\, \sigma_{NN} \rangle = \det T_N[\varphi]$$

$$\varphi(\lambda) = \left(\frac{\alpha - \lambda^{-1}}{\alpha - \lambda} \right)^{1/2} \qquad \alpha = \sinh \frac{2E^v}{kT} \sinh \frac{2E^h}{kT}$$

$$T > T_c \quad \Rightarrow \quad \alpha > 1$$



Wu, McCoy (1966-68)

Observe:

$$G_{\tilde{\Upsilon}}(\lambda) = \begin{pmatrix} 1 & \lambda^{-N}\varphi \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{N}\varphi^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-N}\varphi \end{pmatrix}$$

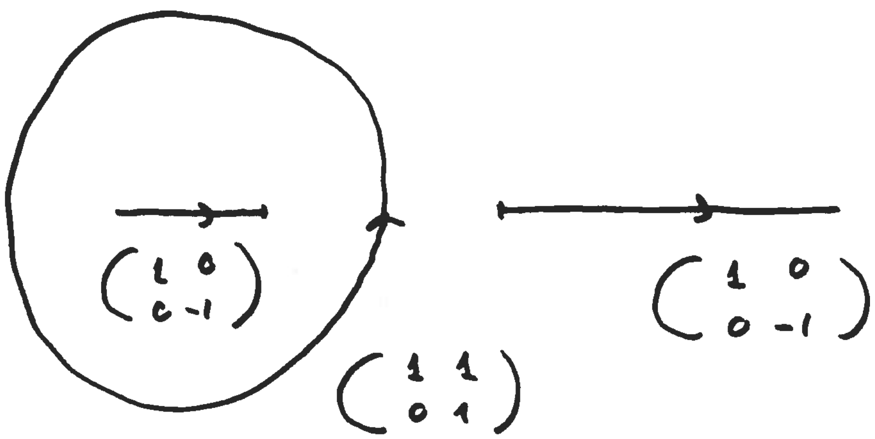
Put

$$\Psi(\lambda) := \tilde{\Upsilon}(\lambda)\,\Psi_0(\lambda)$$

$$\Psi_0(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{N}\varphi^{-1} \end{pmatrix}$$

$$\Downarrow$$

- $\Psi_+(\lambda) = \Psi_-(\lambda)\, G_\Psi \qquad \lambda \in \Gamma_\Psi$



in particular, $G_\Psi \equiv \text{const}$

Observe also,

$$\frac{d\Psi_0}{d\lambda} = \left(\frac{A_1^0}{\lambda} + \frac{A_2^0}{\lambda - a} + \frac{A_3^0}{\lambda - a^{-1}} \right) \Psi_0$$

$$A_1^0 = \left(N + \frac{1}{2}\right) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{2,3}^0 = \pm \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Downarrow \; !$$

$$\frac{d\Psi}{d\lambda} = \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - a} + \frac{A_3}{\lambda - a^{-1}} \right) \Psi$$

$$\frac{d\Psi}{da} = \left(\frac{B_1}{\lambda - a} + \frac{B_2}{\lambda - a^{-1}} \right) \Psi$$

$$\Downarrow$$

$\det T_N(t) =$ τ- function of the sixth Painlevé equation

$$t = a^{-2} \equiv \left(\sinh \frac{2E^v}{kT} \sinh \frac{2E^h}{kT} \right)^{-2}.$$

! Remark

$$\begin{cases} \Psi_\lambda = A(\lambda)\Psi \\ \Psi_a = B(\lambda)\Psi \\ \Psi_{N+1} = C(\lambda)\Psi \end{cases}$$

$$A_a - B_\lambda = [B, A] \quad - \quad \mathrm{P}\underline{\overline{\mathrm{VI}}}$$

$$C_\lambda + C A_N - A_{N+1} C \quad - \quad d\text{-}\mathrm{P}\underline{\overline{\mathrm{VI}}}$$

$$C_a + C B_N - B_{N+c} C \quad - \quad \text{Toda}$$

Appendix 2

Ising model. Double scaling and Painlevé III

(Tracy - McCoy - Wu 77')

$$\tilde{X}(z) := \begin{cases} Y(z)\begin{pmatrix} 1 & z^n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_+^{-1} & 0 \\ 0 & u_+ \end{pmatrix} & |z|<1 \\ \\ Y(z)\begin{pmatrix} 1 & 0 \\ z^{-n} & 1 \end{pmatrix}\begin{pmatrix} u_- & 0 \\ 0 & u_-^{-1} \end{pmatrix} & |z|>1 \end{cases}$$

$$\varphi(z) = u_+ u_-$$

$$\tilde{X}_-(z) = \tilde{X}_+(z)\begin{pmatrix} 0 & -\frac{u_+}{u_-} z^n \\ \frac{u_-}{u_+} z^{-n} & 1 \end{pmatrix}$$

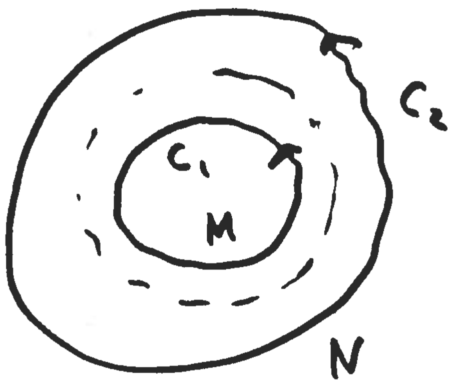
$$\equiv \tilde{X}_+(z)\begin{pmatrix} 1 & -\frac{u_+}{u_-} z^n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{u_-}{u_+} z^{-n} & 1 \end{pmatrix}$$

$$\tilde{X}(z) \to X(z) = \begin{cases} \tilde{X} M & |z|<1 \\ \tilde{X} N^{-1} & |z|>1 \end{cases}$$

$$M = \begin{pmatrix} 1 & -\frac{u_+}{u_-} z^n \\ 0 & 1 \end{pmatrix}$$

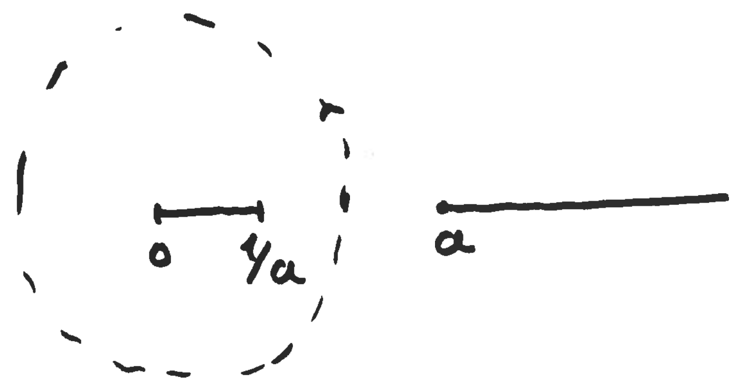
$$N = \begin{pmatrix} 1 & 0 \\ \frac{u_-}{u_+} z^{-n} & 1 \end{pmatrix}$$

$X_- = X_+ G$:



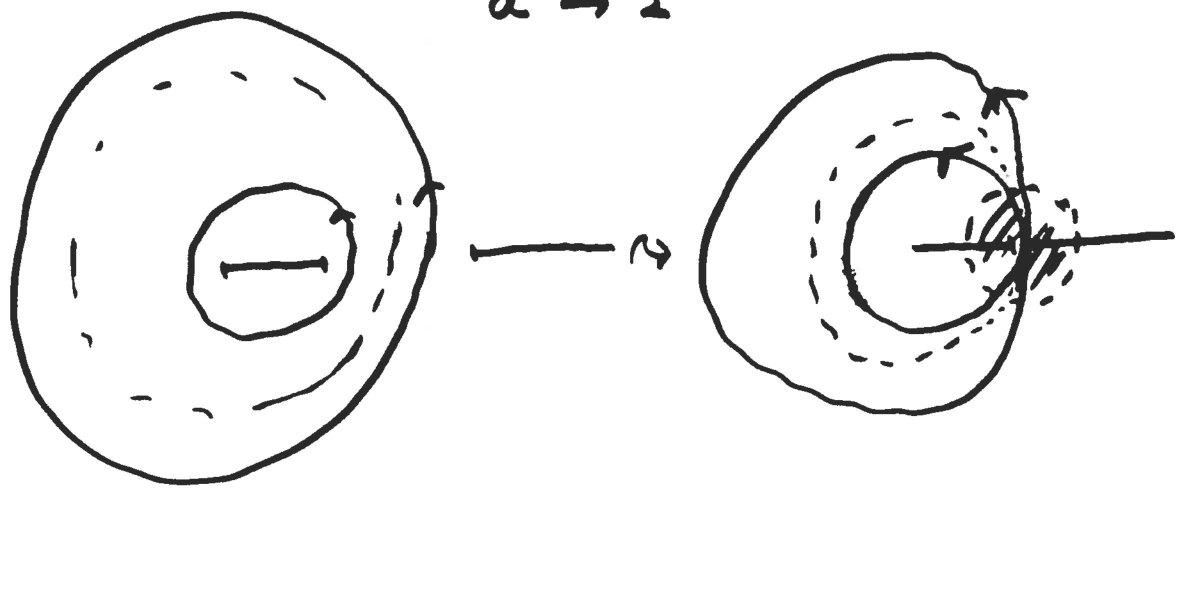
Ising: $\varphi(z) = \left(\dfrac{a - z^{-1}}{a - z}\right)^{1/2}$

$$\equiv \left(\frac{a}{z}\right)^{1/2} \left(\frac{z - a^{-1}}{a - z}\right)^{1/2} \qquad a > 1$$



$$u_+ = \left(\frac{a}{a-z}\right)^{1/2} \qquad u_- = \left(\frac{z - a^{-1}}{z}\right)^{1/2}$$

$a \to 1$



$$a = 1 + \frac{x}{n} \quad , \quad z = 1 + \frac{i\xi}{n}$$

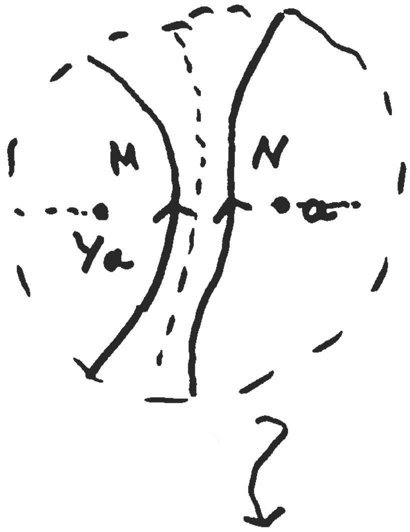
$$z^n \sim e^{i\xi}$$

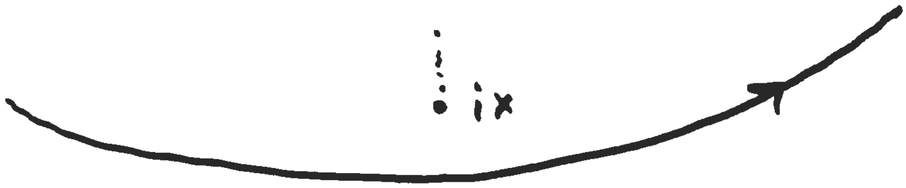
$$u_+ \sim \frac{n^{1/2}}{(x - i\xi)^{1/2}}$$

$$u_- \sim \frac{(i\xi + x)^{1/2}}{n^{1/2}}$$

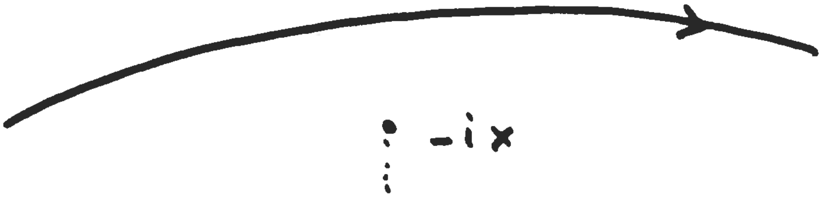
$$M \sim \begin{pmatrix} 1 & -\dfrac{e^{i\xi}}{(x^2+\xi^2)^{1/2}} \\ 0 & 1 \end{pmatrix}$$

$$N \sim \begin{pmatrix} 1 & 0 \\ \dfrac{e^{-i\xi}}{(x^2+\xi^2)^{-1/2}} & 1 \end{pmatrix}$$





$$\begin{pmatrix} 1 & \dfrac{e^{i\xi}}{(\xi^2+x^2)^{1/2}} \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 \\ e^{-i\xi}(\xi^2+x^2)^{1/2} & 1 \end{pmatrix}$$

$$G(\xi) = e^{d(\xi)\sigma_3}\, G_0\, e^{-d(\xi)\sigma_3}$$

$$d(\xi) = \frac{i\xi}{2} - \frac{1}{4}\ln(\xi^2+x^2)$$

$$\boxed{\frac{d}{d\xi} d(\xi) = c_1 + \frac{c_2}{\xi - ix} + \frac{c_3}{\xi+ix} \Rightarrow \overline{\underline{PV}} \to \overline{\underline{PIII}}}$$

$$\overset{\circ}{\Psi}(\xi) := \overset{\circ}{X}(\xi)\, e^{d(\xi)\sigma_3} :$$

- $\overset{\circ}{\Psi}_- = \overset{\circ}{\Psi}_+ G_0$ $\qquad\qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

- $\overset{\circ}{\Psi}(\xi) = \left(I + \frac{m_1}{\xi} + \cdots\right) \times e^{d(\xi)\sigma_3}$ $\qquad\qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$\frac{\partial \overset{\circ}{\Psi}}{\partial \xi} \overset{\circ}{\Psi}{}^{-1} = \frac{i}{2}\sigma_3 + \frac{A_1}{\xi - ix} + \frac{A_2}{\xi + ix} \equiv A(\xi; m_1)$$

$$\frac{\partial \overset{\circ}{\Psi}}{\partial x} \overset{\circ}{\Psi}{}^{-1} = -\frac{iA_1}{\xi - ix} + \frac{iA_2}{\xi + ix} \equiv U(\xi; m_1)$$

$$\Downarrow$$

$$A_x - U_\xi = [U, A] \Longleftrightarrow P\underline{\overline{V}} \quad \text{for}$$

$u(x) \text{ "}{:=}\text{" } (m_1)_{12}$

D.M.