Synchronizing Finite Automata

IV-V. The Road Coloring Problem

Mikhail Volkov

Ural State University, Ekaterinburg, Russia

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1. Recap

Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

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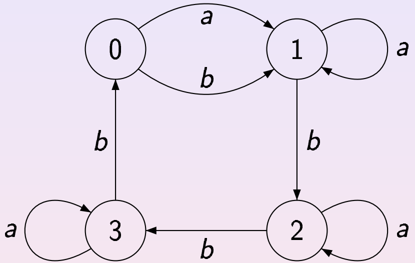

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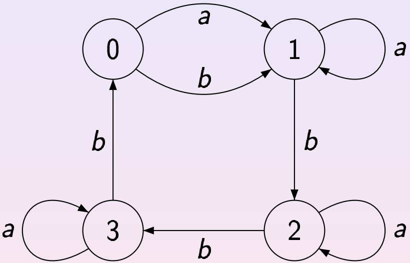

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Studying synchronizing automata, it is natural to restrict to the strongly connected case. For instance, it suffices to prove the Černý conjecture for this case, the general case would be an easy consequence.

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a synchronizing automaton with n states. Consider the set S of all states to which $\mathscr{A}$ can be synchronized and let $m = |S|$. If $q \in S$, then there exists a reset word $w \in \Sigma^*$ such that $Q.w = \{q\}$. For each $a \in \Sigma$, we have $Q.wa = \{\delta(q, a)\}$ whence wa also is a reset word and $\delta(q, a) \in S$. Thus, restricting the function δ to $S \times \Sigma$, we get a subautomaton $\mathscr{S}$ with the state set S. Obviously, $\mathscr{S}$ is synchronizing and strongly connected.

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If the Černý conjecture holds true for strongly connected synchronizing automata, $\mathscr{S}$ has a reset word v of length $(m-1)^2$.

Now consider the partition π of Q into $n-m+1$ classes one of which is S and all others are singletons. Then π is a congruence of the automaton $\mathscr{A}$.

We recall the notion of a congruence and the related notion of the quotient automaton w.r.t. a congruence in the next slide. They will be essentially used in this lecture!

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5. Congruences and Quotient Automata

An equivalence π on the state set Q of a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is called a congruence if $(p, q) \in \pi$ implies $\big(\delta(p, a), \delta(q, a)\big) \in \pi$ for all $p, q \in Q$ and all $a \in \Sigma$. For π being a congruence, $[q]_\pi$ is the π-class containing the state q.

The *quotient* $\mathscr{A}/\pi$ is the DFA $\langle Q/\pi, \Sigma, \delta_\pi \rangle$ where $Q/\pi = \{[q]_\pi \mid q \in Q\}$ and the function δ_π is defined by the rule $\delta_\pi([q]_\pi, a) = [\delta(q, a)]_\pi$.

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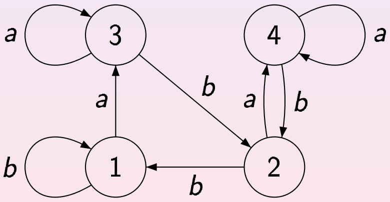

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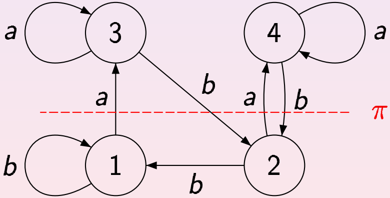

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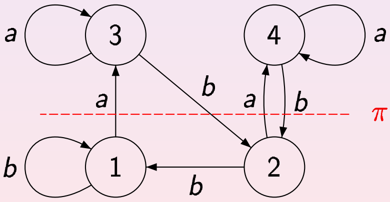

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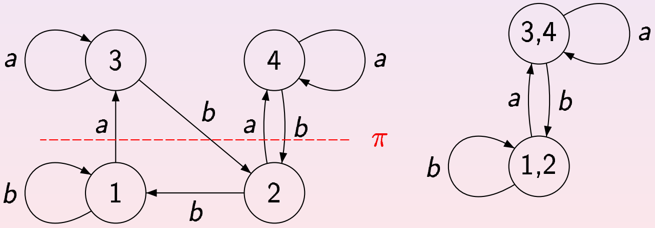

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Return to our reasoning: let π be the partition of Q into $n-m+1$ classes one of which is S and all others are singletons. Then π is a congruence of $\mathscr{A}$.

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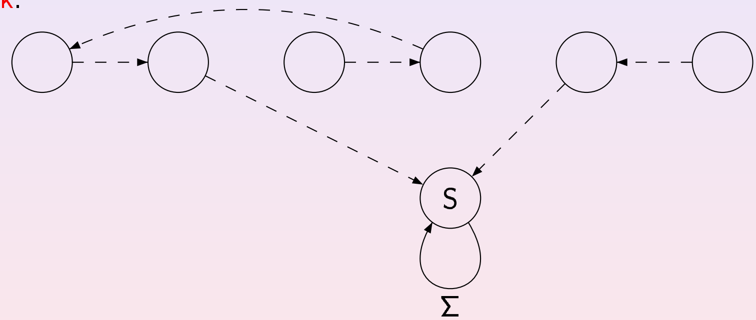

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If a synchronizing automaton with k states has a unique sink, then it has a reset word of length $\leq \frac{k(k-1)}{2}$.

The algorithm makes at most $k-1$ steps and the length of the segment added in the step when t states still hold coins ($k-1 \geq t \geq 1$) is at most $k-t$. The total length is $\leq 1+2+\cdots+(k-1) = \frac{k(k-1)}{2}$.

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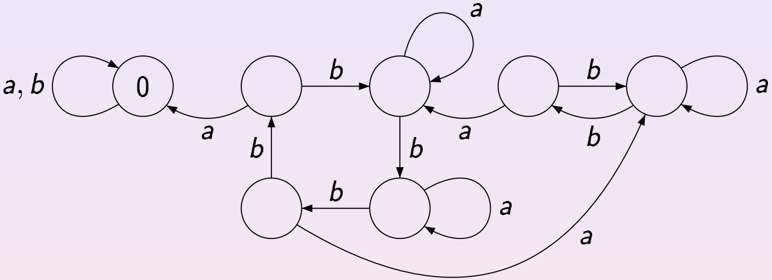

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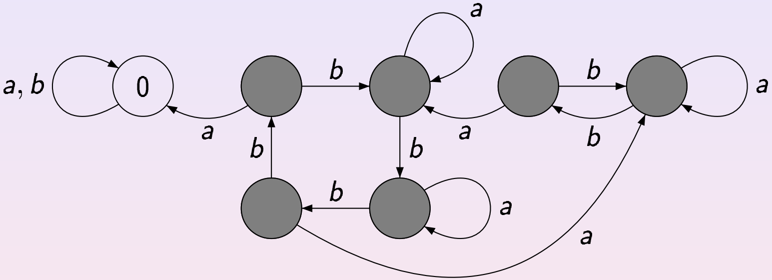

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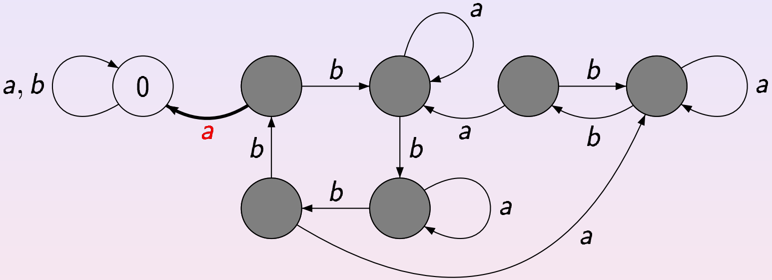

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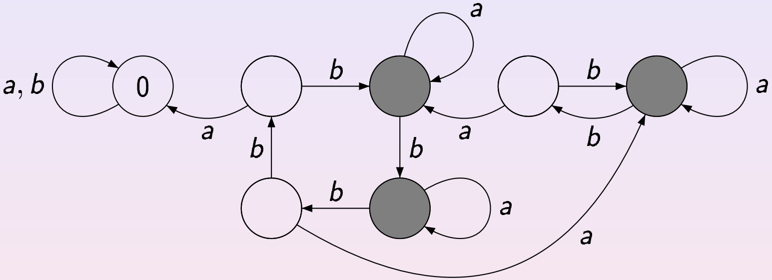

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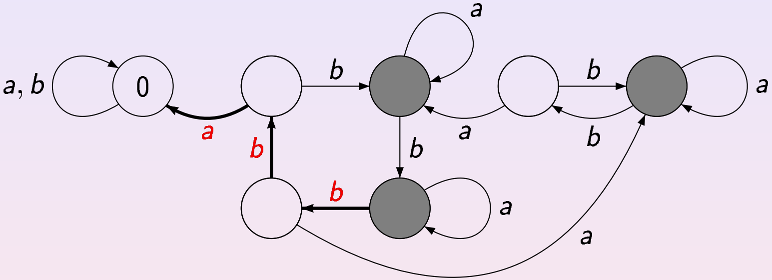

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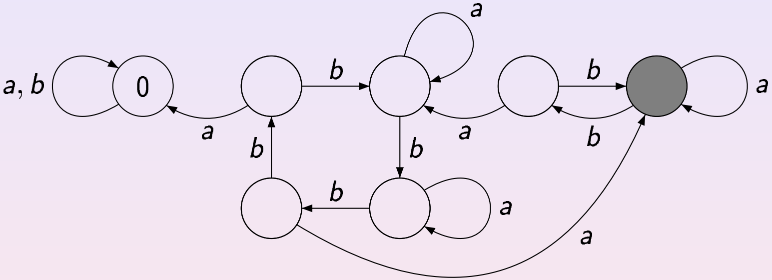

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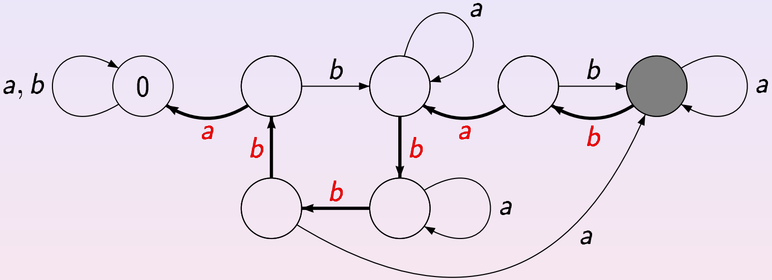

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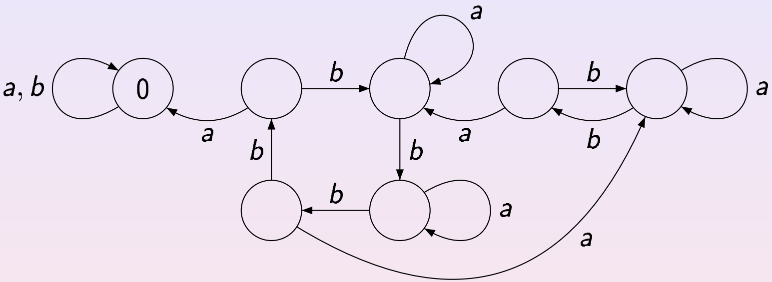

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Return to our reasoning: the quotient $\mathscr{A}/\pi$ is synchronizing with a unique sink and has $n-m+1$ states. Hence, $\mathscr{A}/\pi$ has a reset word u of length $\frac{(n-m+1)(n-m)}{2}$. Then $Q\,.\,u \subseteq S$.

Recall that we have assumed that the automaton $\mathscr{S}$ has a reset word v of length $(m-1)^2$. Then $S.v$ is a singleton, whence also $Q.uv \subseteq S.v$ is a singleton. Thus, uv is a reset word for $\mathscr{A}$, and the length of this word does not exceed

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Observe that such an automaton can be reset to any state. That is, to every state q of the automaton one can assign an instruction (a reset word) w_q such that following w_q one will surely arrive at q from any initial state.

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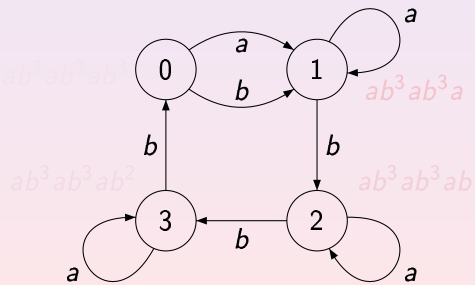

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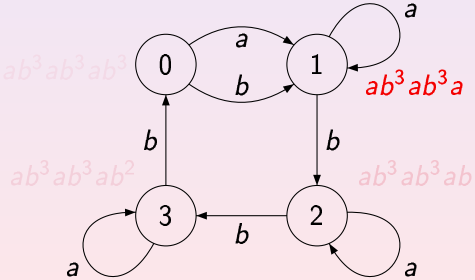

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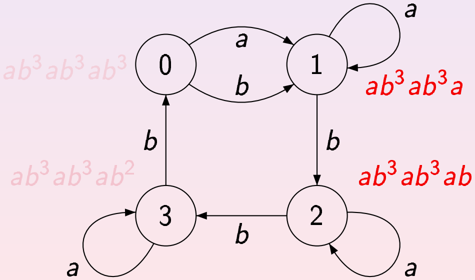

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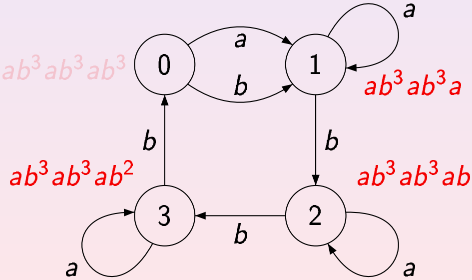

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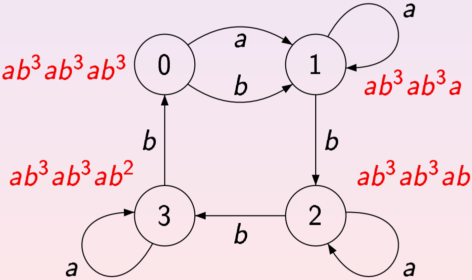

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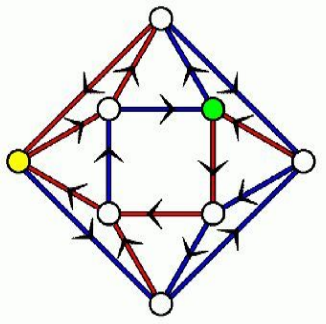
Now think of the automaton as of a scheme of a transport network in which arrows correspond to roads and labels are treated as colors of the roads.



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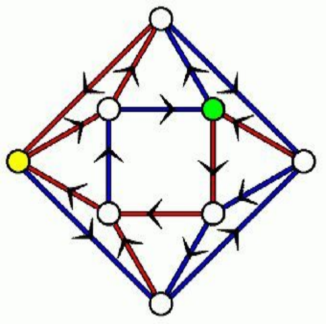

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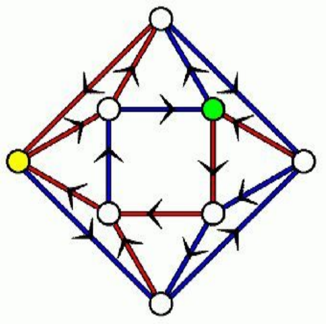

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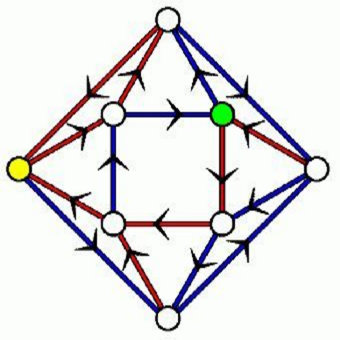

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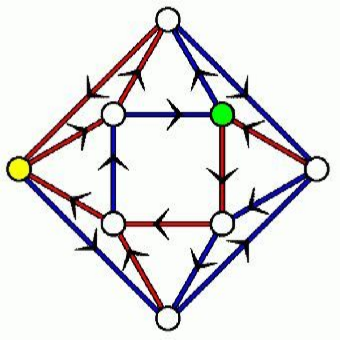

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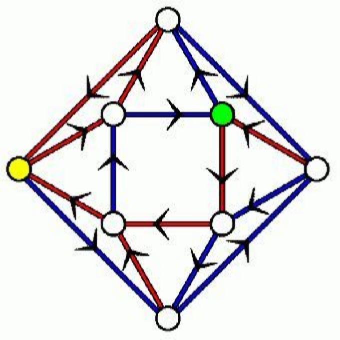

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Now suppose that we have a transport network, that is, a strongly connected digraph.

We aim to help people to orientate in it, and as we have seen, a neat solution may consist in coloring the roads such that our digraph becomes a synchronizing automaton. When is such a coloring possible?

In other words: which strongly connected digraphs may appear as underlying digraphs of synchronizing automata?

An obvious necessary condition:

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Necessity of Primitivity

A less obvious necessary condition is called aperiodicity or primitivity:
the g.c.d. of lengths of all cycles should be equal to 1.

To see why primitivity is necessary, suppose that $\Gamma = (V, E)$ is a strongly connected digraph and $k > 1$ is a common divisor of lengths of its cycles. Take a vertex $v_0 \in V$ and, for $i = 0, 1, \ldots, k-1$, let

$$V_i = \{v \in V \mid \exists \text{ path from } v_0 \text{ to } v \text{ of length } i \pmod{k}\}.$$

Clearly, $V = \bigcup_{i=0}^{k-1} V_i$. We claim that $V_i \cap V_j = \varnothing$ if $i \neq j$.

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Let $v \in V_i \cap V_j$ where $i \neq j$. This means that in Γ there are two paths from v_0 to v: of length $\ell \equiv i \pmod{k}$ and of length $m \equiv j \pmod{k}$.

There is also a path from v to v_0 of length, say, n. Combining it with the two paths above we get a cycle of length $\ell + n$ and a cycle of length $m + n$.

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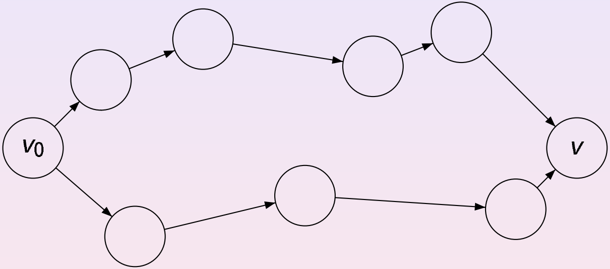

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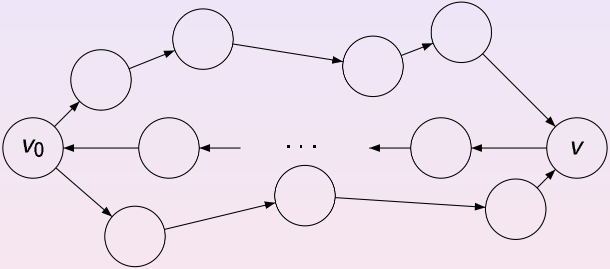

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Thus, V is a disjoint union of $V_0, V_1, \dots, V_{k-1}$, and by the definition each arrow in Γ leads from V_i to $V_{i+1 \pmod{k}}$.

Then Γ definitely cannot be converted into a synchronizing automaton by any labelling of its arrows: for instance, no paths of the same length ℓ originated in V_0 and V_1 can terminate in the same vertex because they end in $V_{\ell \pmod{k}}$ and in $V_{\ell+1 \pmod{k}}$ respectively.

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The Road Coloring Conjecture claims that the two necessary conditions (constant out-degree and primitivity) are in fact sufficient. In other words: *every strongly connected primitive digraph with constant out-degree admits a synchronizing coloring.*

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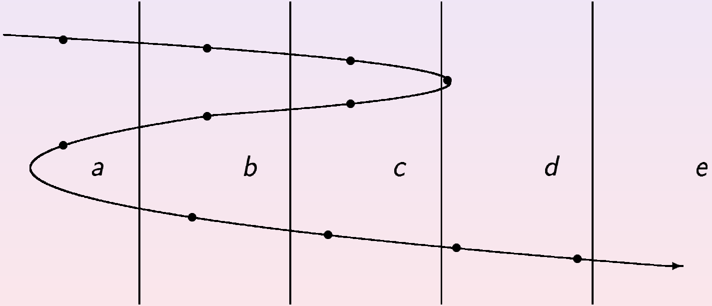

A symbolic trajectory $\ldots abcdcbabcdd \ldots$

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The Road Coloring Conjecture has attracted much attention. There were several interesting partial results, and finally the problem was solved (in the affirmative) in August 2007 by Avraham Trahtman. The solution is published in: The Road Coloring Problem, Israel J. Math. 172 (2009) 51–60. Trahtman's solution got much publicity.

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Trahtman's proof heavily depends on a neat idea of stability which is due to Karel Culik II, Juhani Karhumäki and Jarkko Kari (A note on synchronized automata and Road Coloring Problem, Int. J. Found. Comput. Sci., 13 (2002) 459–471). Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. We define the relation $\sim$ on Q as follows:

$$q \sim q' \iff \forall u \in \Sigma^* \ \exists v \in \Sigma^* \ q\,.\,uv = q'.uv.$$

$\sim$ is called the *stability relation* and any pair (q, q') such that $q \sim q'$ is called *stable*. It is immediate that $\sim$ is a congruence of the automaton $\mathscr{A}$. Also observe that $\mathscr{A}$ is synchronizing iff all pairs are stable.

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Proposition CKK. *Suppose every strongly connected primitive digraph with constant out-degree and more than 1 vertex has a stable coloring. Then the Road Coloring Conjecture holds true.*

The proof is rather straightforward: one inducts on the number of vertices in the digraph. If Γ admits a stable coloring and $\mathscr{A}$ is the resulting automaton, then the quotient automaton $\mathscr{A}/\!\sim$ admits a synchronizing recoloring by the induction assumption.
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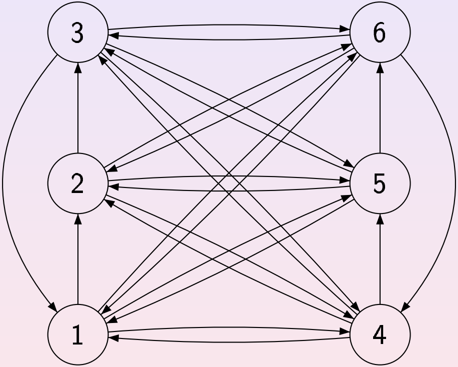
21. Example

Look at the following digraph Γ and one of its colorings. It is not synchronizing (the states 1 and 4 cannot be synchronized).



On can see that the stability relation is the partition 123 | 456.

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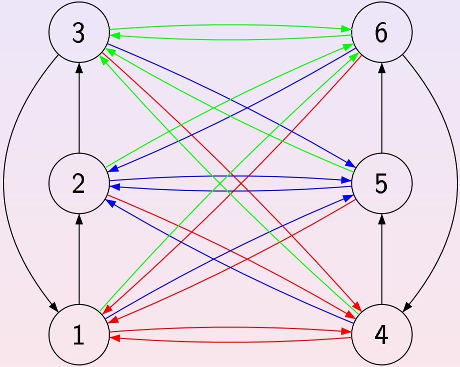

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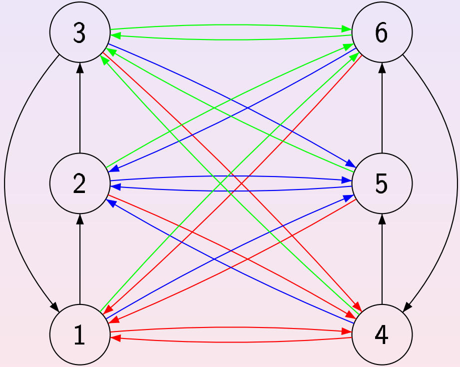

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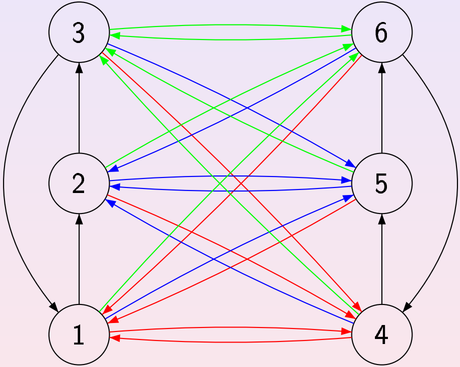

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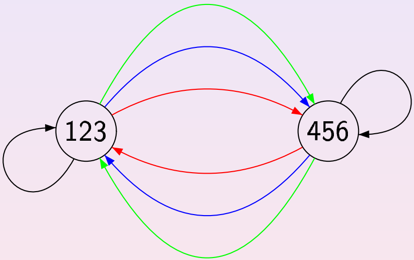

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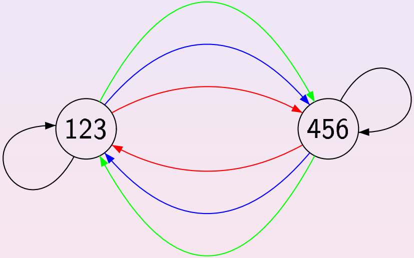

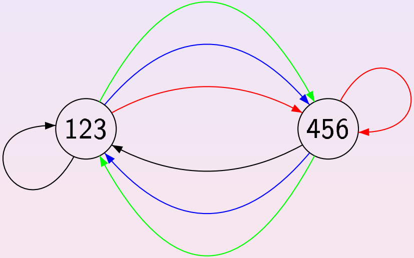

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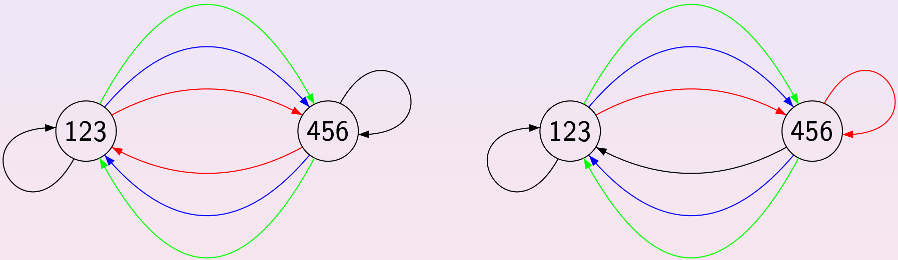

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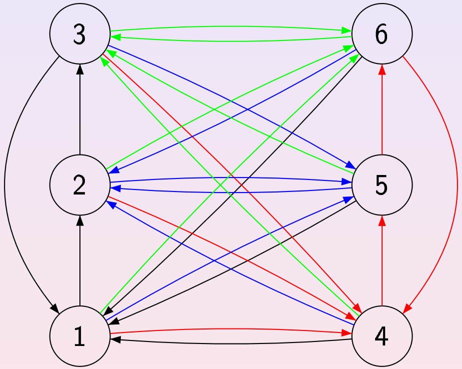

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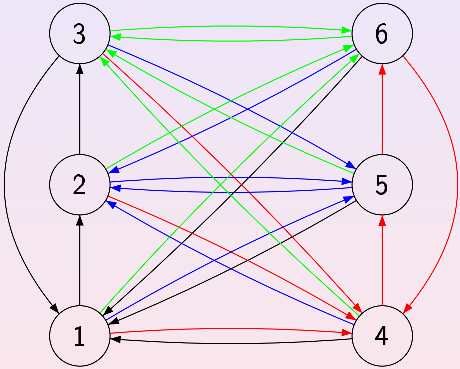

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The proof is clever but not too difficult. One argues by contradiction and studies a hypothetical strongly connected primitive digraph Γ with constant out-degree (*admissible digraph*, for short) and more than 1 vertex such that Γ has no stable coloring.

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Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA. A pair (p, q) of distinct states is a deadlock if $\forall w \in \Sigma^* \; p \,.\, w \neq q \,.\, w$. If an automaton is not synchronizing, it must have deadlocks!

Moreover, if a pair (p, q) is not stable, then for some word $u \in \Sigma^*$ the pair $(p \,.\, u, q \,.\, u)$ is a deadlock.

A clique F is any subset of Q of maximum cardinality such that every pair of states in F is a deadlock.

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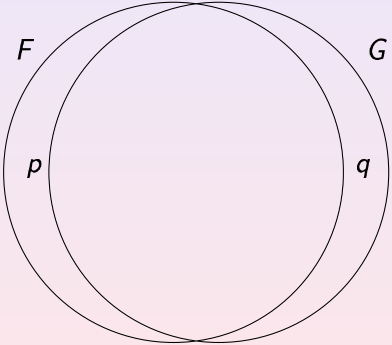

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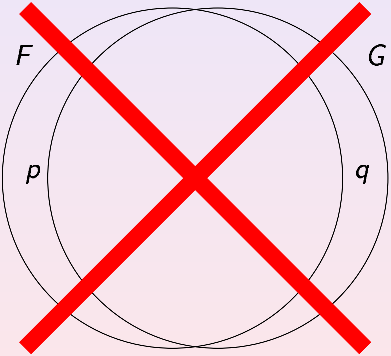

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28. Levels w.r.t. a Letter

Let $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ be a DFA, $a \in \Sigma$. We want to assign to its states a parameter called the level w.r.t. a.

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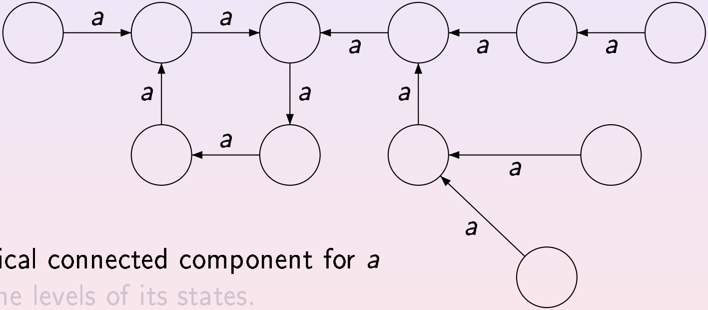

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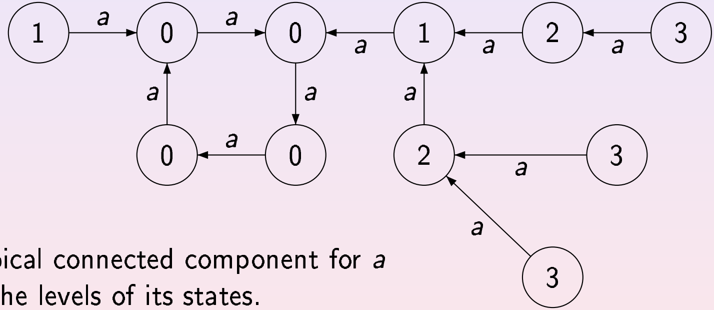

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Proof. Let M be the set of all states of level L w.r.t a. Then $p \,.\, a^L = q \,.\, a^L$ for all $p, q \in M$ whence no pair of states from M forms a deadlock. Thus, if $C \subseteq Q$ is a clique then $|C \cap M| \leq 1$. Take a clique C such that $|C \cap M| = 1$ (it exists since $\mathscr{A}$ is strongly connected.) Then $F = C.a^{L-1}$ is a clique that has all its states except one in the a-cycles. If m is the l.c.m. of the lengths of all a-cycles, $r \,.\, a^m = r$ for any r in any a-cycle. Hence $G = F \,.\, a^m$ is a clique such that

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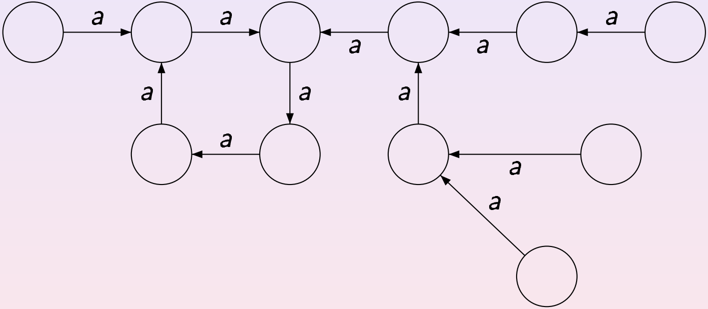

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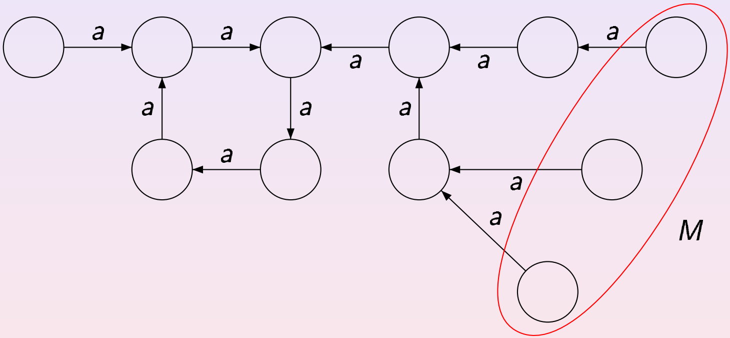

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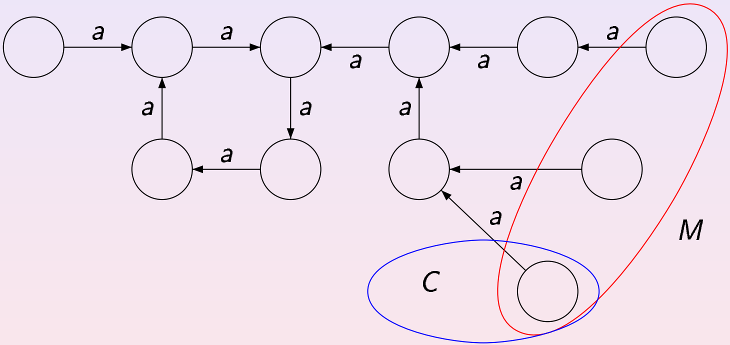

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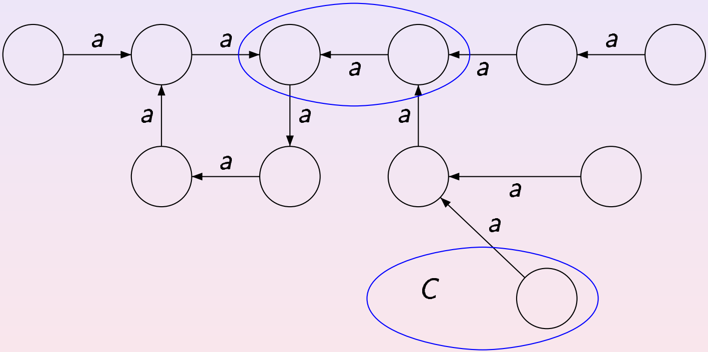

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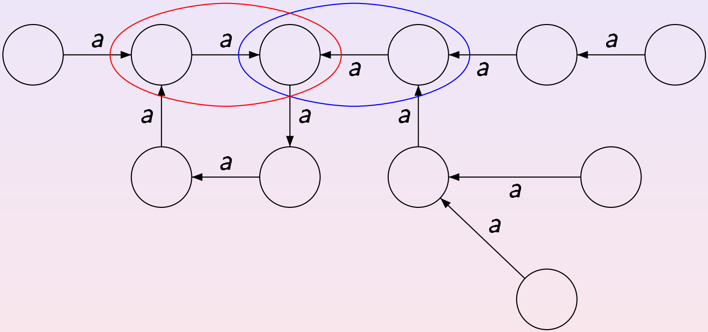

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In order to show that every admissible digraph Γ can be colored such that all states of maximal level w.r.t. a certain letter belong to the same tree, we start with an arbitrary coloring of Γ, take an arbitrary color (=letter) a, and induct on the number N of states that do not lie on any a-cycle in the initial coloring.

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32. Induction Basis

Suppose that $N = 0$. This means that all states lie on a-cycles.

We say that a vertex p of Γ is a bunch if all edges that begin at p lead to the same vertex q.

If all vertices in Γ are bunches, then there is just one a-cycle (since Γ is strongly connected) and all cycles in Γ have the same length. This contradicts the assumption that Γ is primitive.

It is quite interesting that this is the only place in the whole proof where the primitivity condition is invoked.

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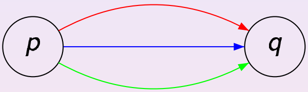

If all vertices in Γ are bunches, then there is just one a-cycle (since Γ is strongly connected) and all cycles in Γ have the same length. This contradicts the assumption that Γ is primitive.
It is quite interesting that this is the only place in the whole proof where the primitivity condition is invoked.

32. Induction Basis

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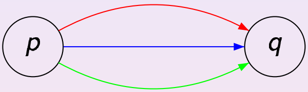

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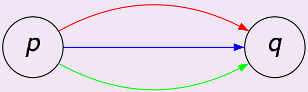

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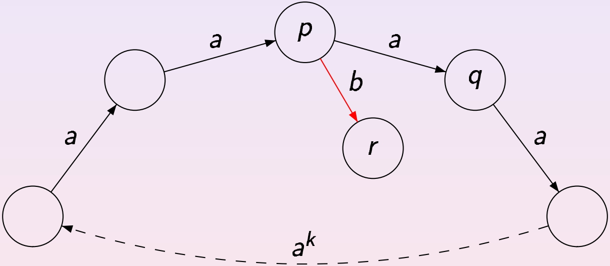
Thus, let p be a state which is not a bunch, let $q = p\,.\,a$ and let $b \neq a$ be such that $r = p\,.\,b \neq q$. We exchange the labels of the edges $p \to q$ and $p \to r$.



It is clear that in the new coloring there is only one state of maximal level w.r.t. a, namely q. Thus, the induction basis is verified.

33. Induction Basis

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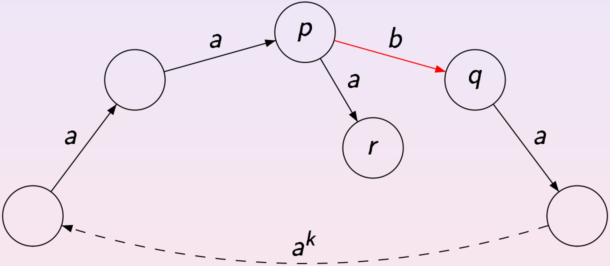

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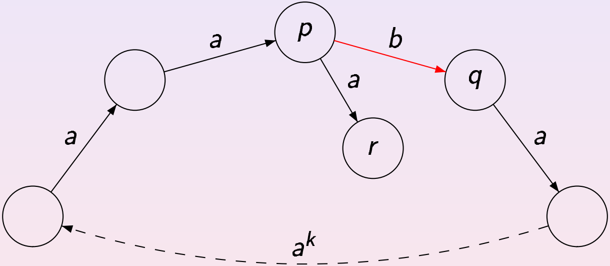

It is clear that in the new coloring there is only one state of maximal level w.r.t. a, namely q. Thus, the induction basis is verified.

Now let $N > 0$. We denote by L the maximum level of the states w.r.t. a in the initial coloring. Observe that $N > 0$ implies $L > 0$. Let p be a state of level L. Since Γ is strongly connected, there is an edge $p' \to p$ with $p' \neq p$, and by the choice of p, the label of this edge is $b \neq a$. Let $p' = s\,.\,a$. One has $t \neq p$. Let $r = p\,.\,a^L$ and let C be the a-cycle on which r lies.

The following considerations split in several cases. In each case except one we can recolor Γ by flipping the labels of two edges such the new coloring either fulfils the property we aim at (all states of maximal level w.r.t. a belong to the same tree) or has more states on the a-cycles (and the induction assumption applies). The remaining case is shown to lead to a contradiction.

34. Induction Step

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