FEDOR V. FOMIN

Parameterized Algorithms I

BERGEN

St. Petersburg, 2011

Classical complexity

A brief review:

- We usually aim for **polynomial-time** algorithms: the running time is $O(n^c)$, where n is the input size.
- Classical polynomial-time algorithms: shortest path, mathching, minimum spanning tree, 2SAT, convext hull, planar drawing, linear programming, etc.
- It is unlikely that polynomial-time algorithms exist for **NP-hard** problems.

Classical complexity

- Unfortunately, many problems of interest are NP-hard: Hamiltonian cycle, 3-coloring, 3SAT, etc.
- We expect that these problems can be solved only in exponential time (i.e., c^n).

Can we say anything nontrivial about NP-hard problems?

Parameterized complexity

Main idea: Instead of expressing the running time as a function $T(n)$ of n, we express it as a function $T(n, k)$ of the input size n and some parameter k of the input.

In other words: we do not want to be efficient on all inputs of size n, only for those where k is small.

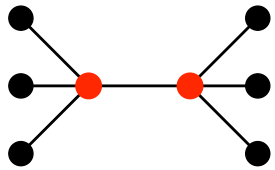
Parameterized complexity

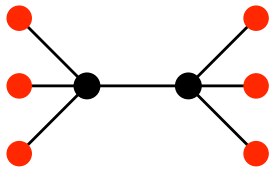
What can be the parameter k?

- The size k of the solution we are looking for.
- The maximum degree of the input graph.
- The diameter of the input graph.
- The length of clauses in the input Boolean formula.
- ...

Parameterized complexity

Problem:	Min Vertex Cover	Max Independent Set
Input:	Graph G, integer k	Graph G, integer k
Question:	Is it possible to cover the edges with k vertices?	Is it possible to find k independent vertices?





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	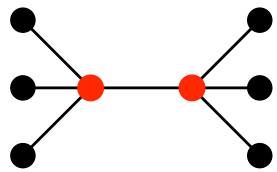	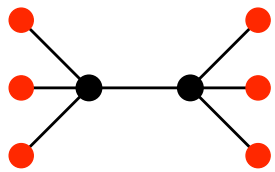
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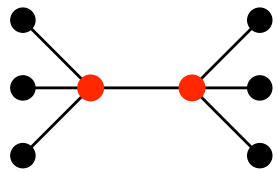
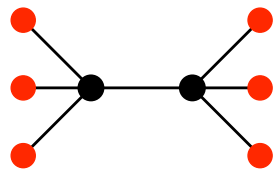
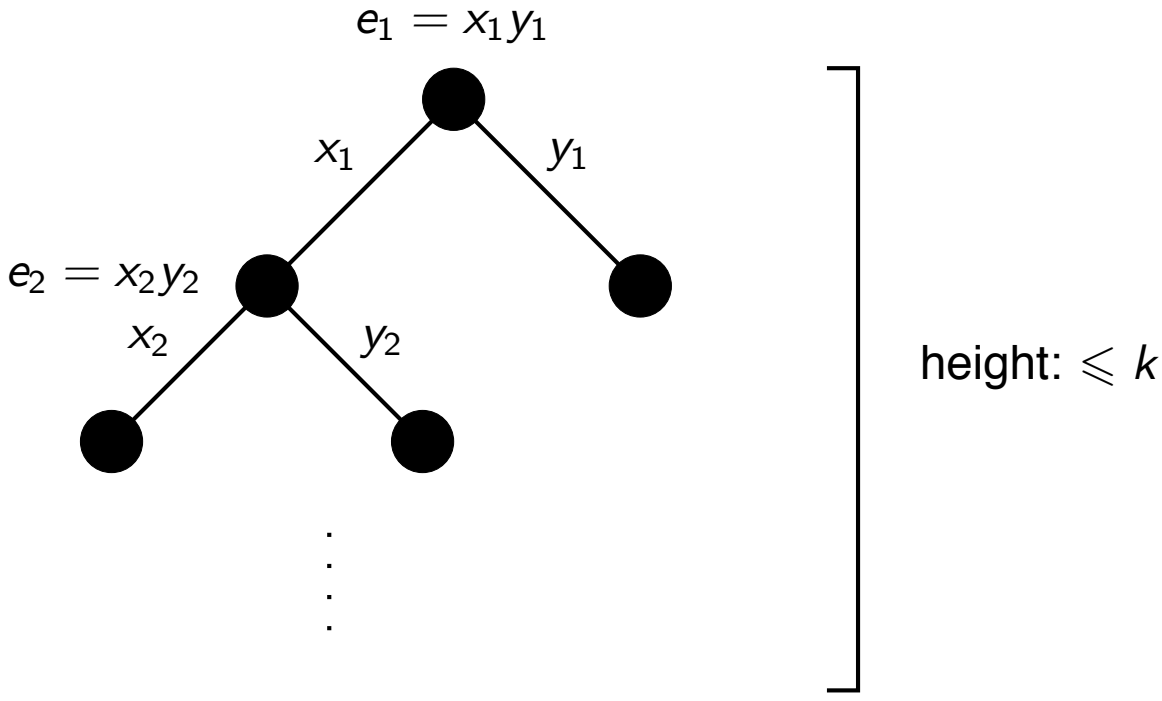
		
Complexity:	NP-complete	NP-complete
Complete	$O(n^k)$ possibilities	$O(n^k)$ possibilities
enumeration:	$O(2^k n^2)$ algorithm exists 	No $n^{o(k)}$ algorithm known 

Bounded search tree method

Algorithm for Minimum Vertex Cover:



Height of the search tree is $\leq k \Rightarrow$ number of leaves is $\leq 2^k \Rightarrow$ complete search requires $2^k \cdot$ poly steps.

Exercise

Problem: Minimum Dominating Set
Input: Graph G, integer k
Question: Is it possible to cover all vertices with k vertices?

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Fixed-parameter tractability

Definition: A *parameterization* of a decision problem is a function that assigns an integer parameter k to each input instance x.

The parameter can be

- explicit in the input (for example, if the parameter is the integer k appearing in the input (G, k) of $\textsc{Vertex Cover}$), or
- implicit in the input (for example, if the parameter is the diameter d of the input graph G).

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Main definition:

A parameterized problem is **fixed-parameter tractable (FPT)** if there is an $f(k)n^c$ time algorithm for some constant c.

Fixed-parameter tractability

Example: Minimum Vertex Cover parameterized by the required size k is FPT: we have seen that it can be solved in time $O(2^k + n^2)$.

Better algorithms are known: e.g, $O(1.2832^k k + k|V|)$.

Main goal of parameterized complexity: to find FPT problems.

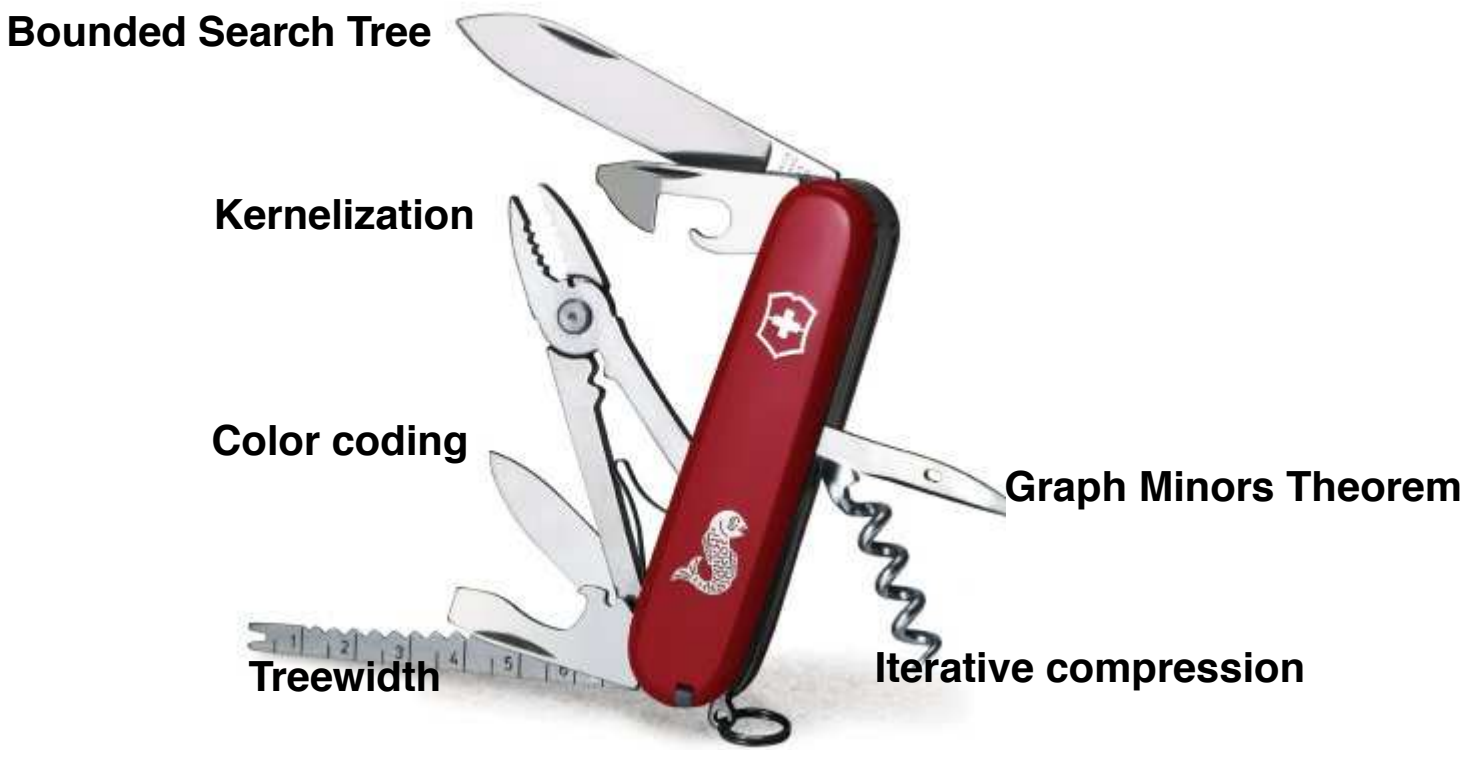
FPT problems

Examples of NP-hard problems that are FPT:

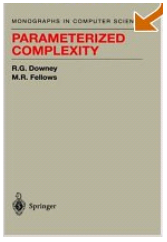
- Finding a vertex cover of size k.
- Finding a path of length k.
- Finding k disjoint triangles.
- Drawing the graph in the plane with k edge crossings.
- Finding disjoint paths that connect k pairs of points.
- ...

FPT algorithmic techniques

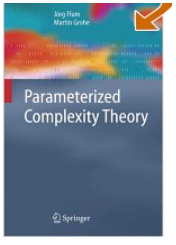
- Significant advances in the past 20 years or so (especially in recent years).
- Powerful toolbox for designing FPT algorithms:



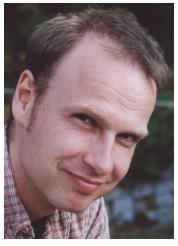
Books

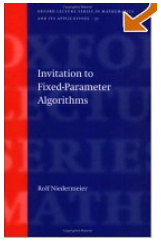


Downey-Fellows: Parameterized Complexity, Springer, 1999



Flum-Grohe: Parameterized Complexity Theory, Springer, 2006





Niedermeier: Invitation to Fixed-Parameter Algorithms, Oxford University Press, 2006.

Goals of the course

- Demonstrate techniques that were successfully used in the analysis of parameterized problems.
  - Determine quickly if a problem is FPT.
  - Design fast algorithms (improve the function $f(k)$).

Notes

This course and slides are based on Dániel Marx's lectures on parameterized complexity
`http://www.cs.bme.hu/~dmarx/`

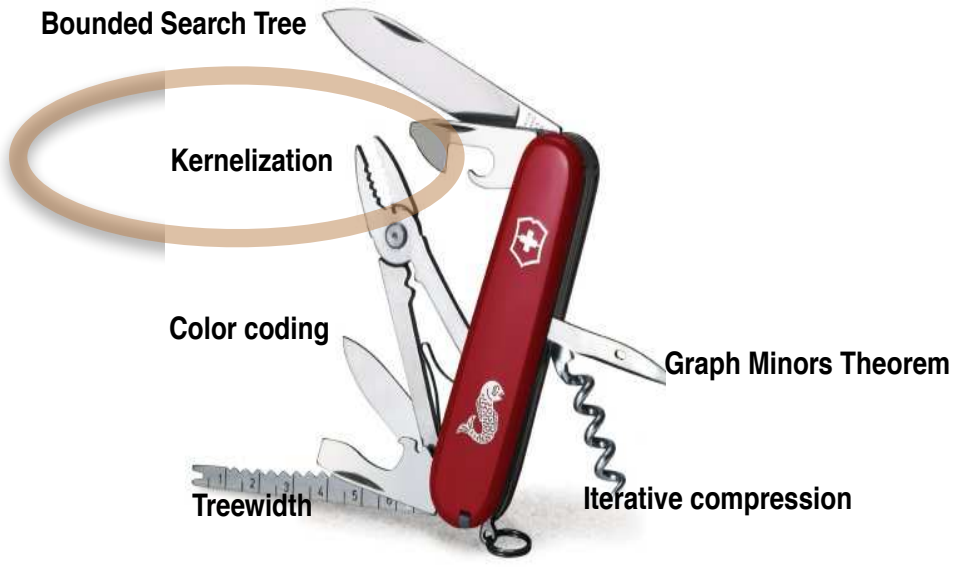
What is missing but worth to discuss

- **Complexity:**
  - W-hierarchy, reductions,
  - Exponential Time Hypothesis and lower bounds for FPT algorithms
  - ETH and parameterized complexity
  - Polynomial Time Hierarchy and lower bounds on the sizes of kernels
- **Algorithms:**
  - Integer programming
  - Meta theorems: First Order Logic, Finite Integer Index
  - Multi-cut algorithms and Directed FVS

Notes

- **Warning:** The results presented for particular problems are not necessarily the best known results or the most useful approaches for these problems.
- Conventions:
  - Unless noted otherwise, k is the parameter.
  - O^* notation: $O^*(f(k))$ means $O(f(k) \cdot n^c)$ for some constant c.
  - Citations are mostly omitted (only for classical results).
  - We gloss over the difference between decision and search problems.

Kernelization



Introduction and definition

Part I: Crown Decomposition

Part II: Sunflower Lemma

Kernelization

Definition: *Kernelization* is a polynomial-time transformation that maps an instance (I, k) to an instance (I', k') such that

- (I, k) is a yes-instance if and only if (I', k') is a yes-instance,
- $k' \leq k$, and
- $|I'| \leq f(k)$ for some function $f(k)$.

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Simple fact: If a problem has a kernelization algorithm, then it is FPT.

Proof: Solve the instance (I', k') by brute force.

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Simple fact: If a problem has a kernelization algorithm, then it is FPT.
Proof: Solve the instance (I', k') by brute force.

Converse: Every FPT problem has a kernelization algorithm.
Proof: Suppose there is an $f(k)n^c$ algorithm for the problem.

- If $f(k) \leq n$, then solve the instance in time $f(k)n^c \leq n^{c+1}$, and output a trivial yes- or no-instance.
- If $n < f(k)$, then we are done: a kernel of size $f(k)$ is obtained.

Kernelization for Vertex Cover

General strategy: We devise a list of reduction rules, and show that if none of the rules can be applied and the size of the instance is still larger than $f(k)$, then the answer is trivial.

Reduction rules for Vertex Cover instance (G, k):

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$
Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k-1)$

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Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k-1)$

If neither Rule 1 nor Rule 2 can be applied:

- If $|V(G)| > k(k+1) \Rightarrow$ There is no solution (every vertex should be the neighbor of at least one vertex of the cover).
- Otherwise, $|V(G)| \leq k(k+1)$ and we have a $k(k+1)$ vertex kernel.

Kernelization for Vertex Cover

Let us add a third rule:

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$
Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k-1)$
Rule 3: If $d(v) = 1$, then we can assume that its neighbor u is in the solution $\Rightarrow (G \setminus (u \cup v), k-1)$.

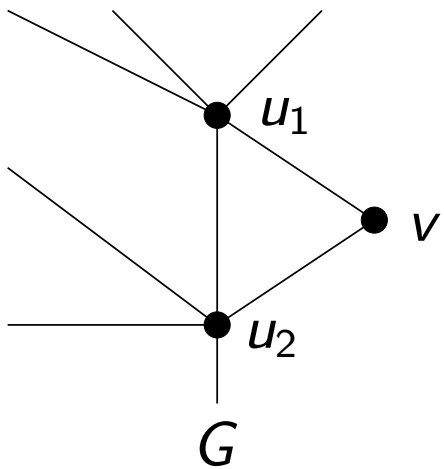
If none of the rules can be applied, then every vertex has degree at least 2.
$\Rightarrow |V(G)| \leq |E(G)|$

- If $|E(G)| > k^2 \Rightarrow$ There is no solution (each vertex of the solution can cover at most k edges).
- Otherwise, $|V(G)| \leq |E(G)| \leq k^2$ and we have a k^2 vertex kernel.

Kernelization for Vertex Cover

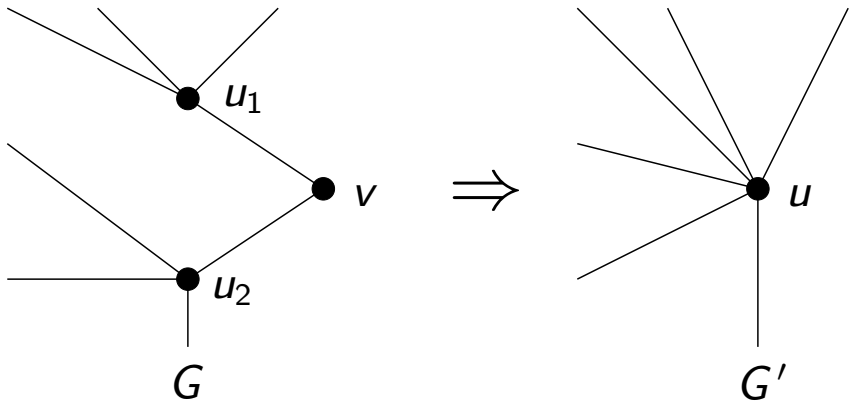
Let us add a fourth rule:

Rule 4a: If v has degree 2, and its neighbors u_1 and u_2 are adjacent, then we can assume that u_1, u_2 are in the solution $\Rightarrow$ $(G \setminus \{u_1, u_2, v\}, k-2)$.



Kernelization for Vertex Cover

Rule 4b: If v has degree 2, then G' is obtained by identifying the two neighbors of v and deleting $v \Rightarrow (G', k-1)$.



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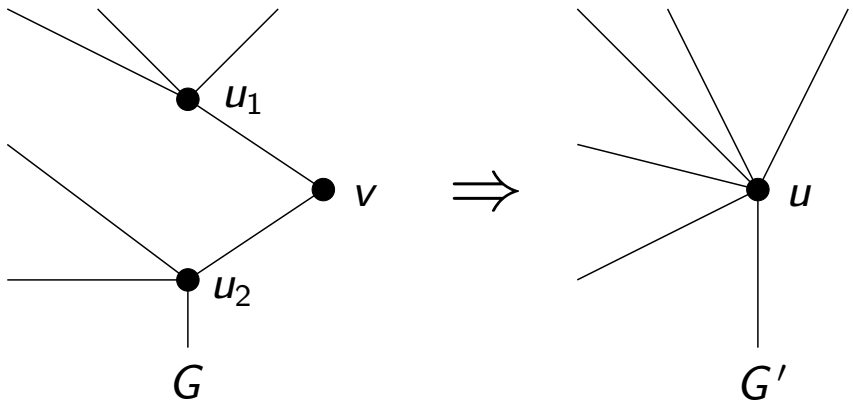

Correctness:

- Let S' be a vertex cover of size $k-1$ for G'.
- If $u \in S \Rightarrow (S' \setminus u) \cup \{u_1, u_2\}$ is a vertex cover of size k for G.
- If $u \notin S \Rightarrow S' \cup v$ is a vertex cover of size k for G.

Kernelization for Vertex Cover

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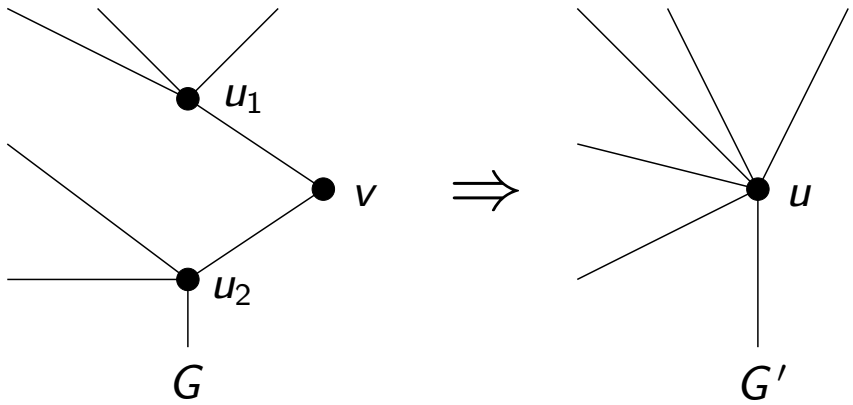

Correctness:

- Let S be a vertex cover of size k for G.
- If $u_1, u_2 \in S \Rightarrow (S \setminus \{u_1, u_2, v\}) \cup u$ is a vertex cover of size $k-1$ for G'.
- If exactly one of u_1 and u_2 is in S, then $v \in S \Rightarrow$ $(S \setminus \{u_1, u_2, v\}) \cup u$ is a vertex cover of size $k-1$ for G'.
- If $u_1, u_2 \notin S$, then $v \in S \Rightarrow (S \setminus v)$ is a vertex cover of size $k-1$ for G'.

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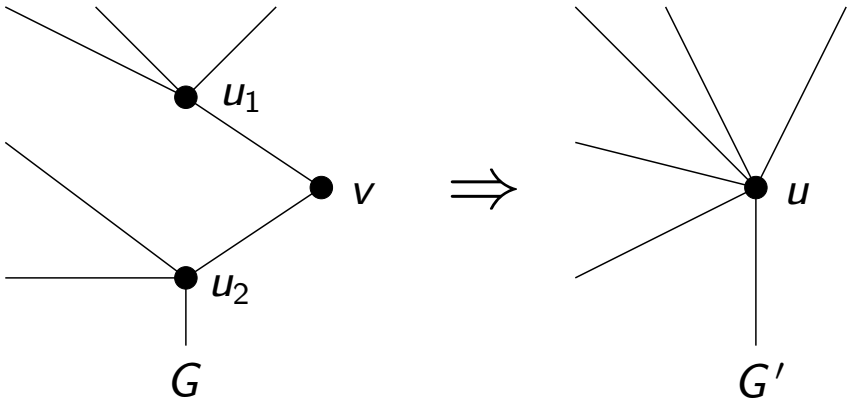
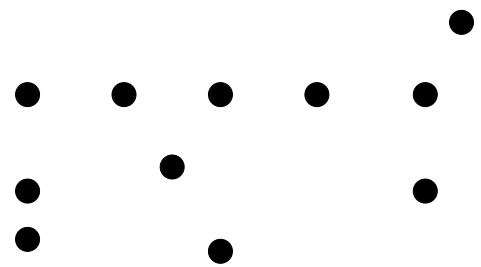


Kernel size:

- If $|E(G)| > k^2 \Rightarrow$ There is no solution (each vertex of the solution can cover at most k edges).
- Otherwise, $|V(G)| \leq 2|E(G)|/3 \leq \frac{2}{3}k^2$ and we have a $\frac{2}{3}k^2$ vertex kernel.

Covering Points with Lines

Task: Given a set P of n points in the plane and an integer k, find k lines that cover all the points.



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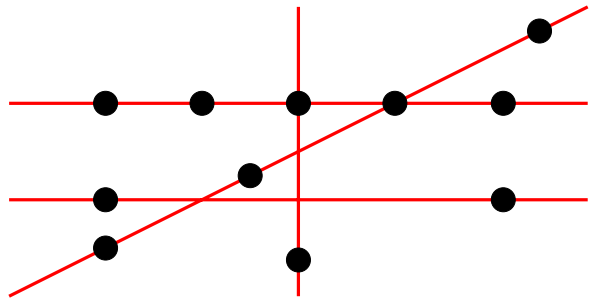

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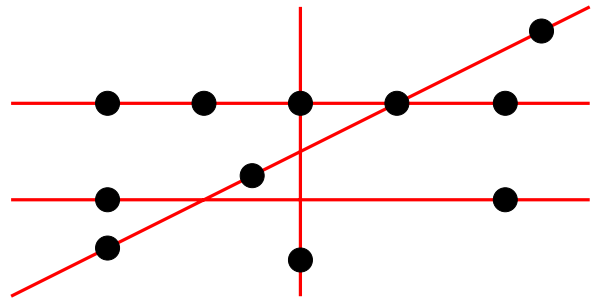

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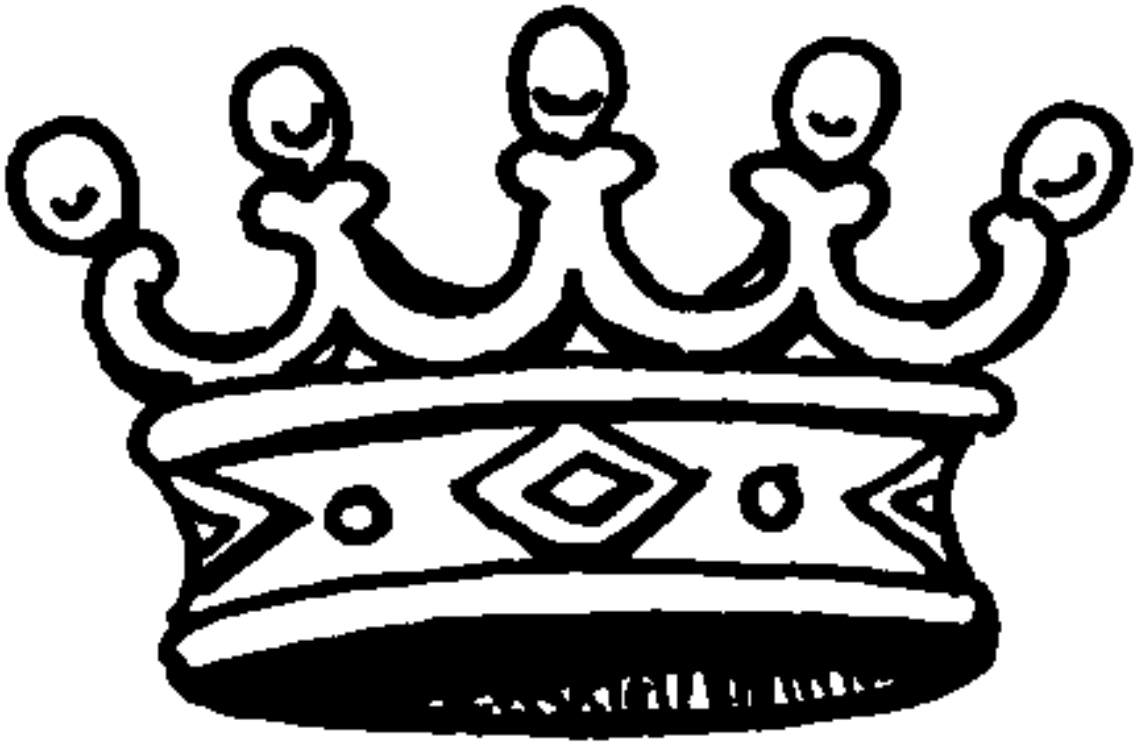
Reduction Rule:
If a candidate line covers a set S of more than k points $\Rightarrow$ $(P \setminus S, k-1)$.

If this rule cannot be applied and there are still more than k^2 points, then there is no solution $\Rightarrow$ Kernel with at most k^2 points.

Kernelization

- Kernelization can be thought of as a polynomial-time preprocessing before attacking the problem with whatever method we have. "It does no harm" to try kernelization.
- Some kernelizations use lots of simple reduction rules and require a complicated analysis to bound the kernel size...
- ... while other kernelizations are based on surprising nice tricks (Next: Crown Reduction and the Sunflower Lemma).
- Possibility to prove lower bounds.

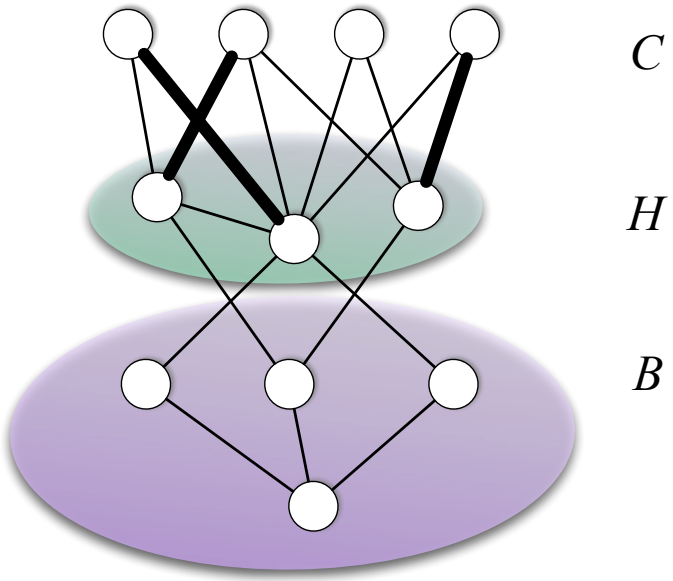
Crown Reduction



Crown Reduction

Definition: A **crown decomposition** is a partition $C \cup H \cup B$ of the vertices such that

- C is an independent set,
- there is no edge between C and B,
- there is a matching between C and H that covers H.



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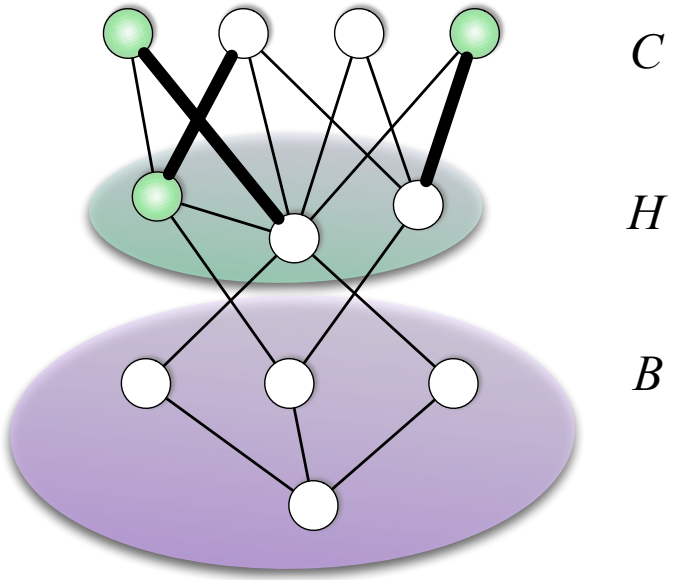

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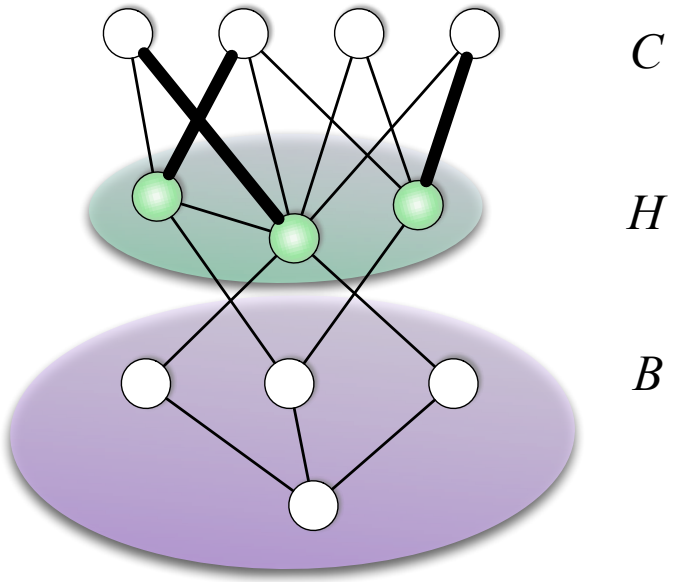

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$\Rightarrow (G \setminus (H \cup C), k - |H|)$.

Crown Reduction

Key lemma:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

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This gives a $3k$ vertex kernel for Vertex Cover.

Proof

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$,
- find a crown decomposition,
- or conclude that the graph has at most $3k$ vertices.

For the proof, we need the classical Kőnig's Theorem.

$\tau(G)$: size of the minimum vertex cover
$\nu(G)$: size of the maximum matching (independent set of edges)

Theorem: [Kőnig, 1931] If G is **bipartite,** then

$$\tau(G) = \nu(G)$$

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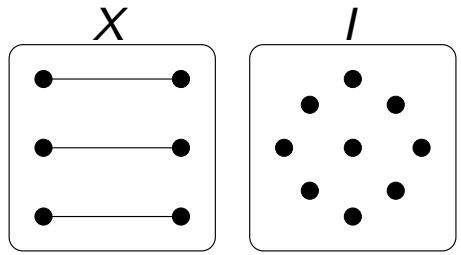
Find (greedily) a maximal matching; if its size is at least $k+1$, then we are done. The rest of the graph is an independent set I.



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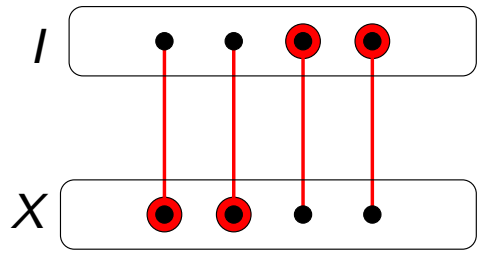
Find a maximum matching/minimum vertex cover in the bipartite graph between X and I.



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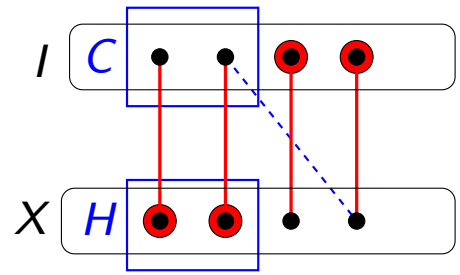
Proof:

Case 1: The minimum vertex cover contains at least one vertex of X

$\Rightarrow$ There is a crown decomposition.



Proof

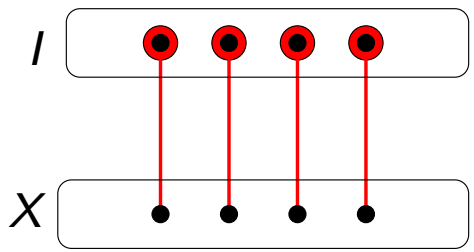
Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$,
- find a crown decomposition,
- or conclude that the graph has at most $3k$ vertices.

Proof:

Case 1: The minimum vertex cover contains at least one vertex of X
$\Rightarrow$ There is a crown decomposition.

Case 2: The minimum vertex cover contains only vertices of I $\Rightarrow$ It contains every vertex of I
$\Rightarrow$ There are at most $2k+k$ vertices.



Dual of Vertex Coloring

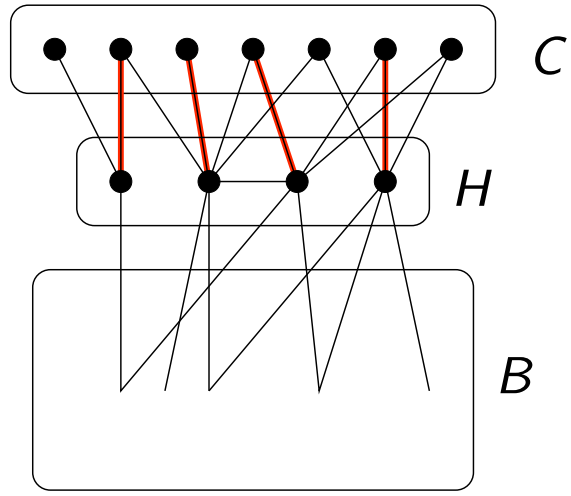
Parameteric dual of k-Coloring. Also known as Saving k Colors.

Task: Given a graph G and an integer k, find a vertex coloring with $|V(G)| - k$ colors.

Crown rule for Dual of Vertex Coloring:

Suppose there is a crown decomposition for the **complement graph** $\overline{G}$.

- C is a clique in G: each vertex needs a distinct color.
- Because of the matching, it is possible to color H using only these $|C|$ colors.
- These colors cannot be used for B.
- $(G \setminus (H \cup C), k - |H|)$



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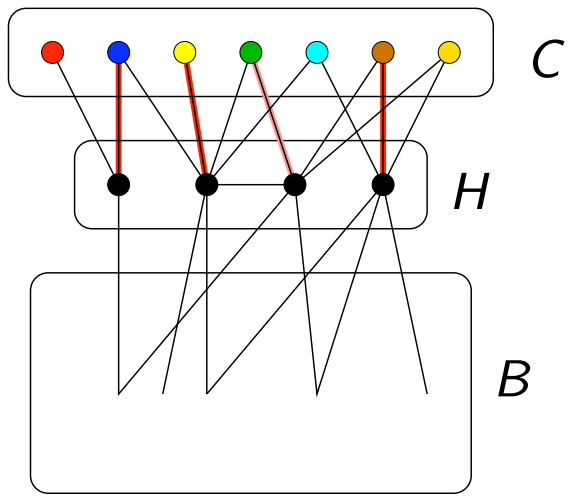

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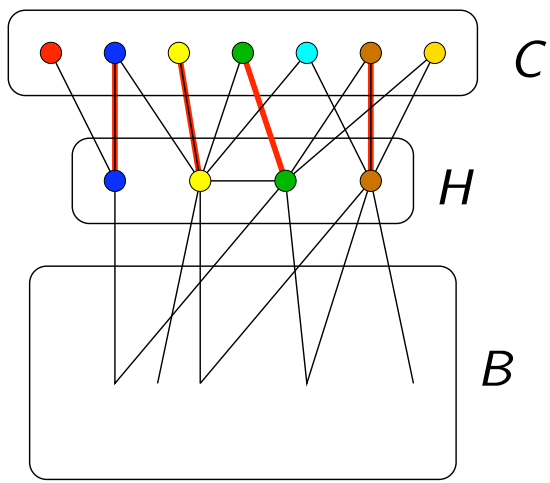


Crown Reduction for Dual of Vertex Coloring

Use the key lemma for the complement $\overline{G}$ of G:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

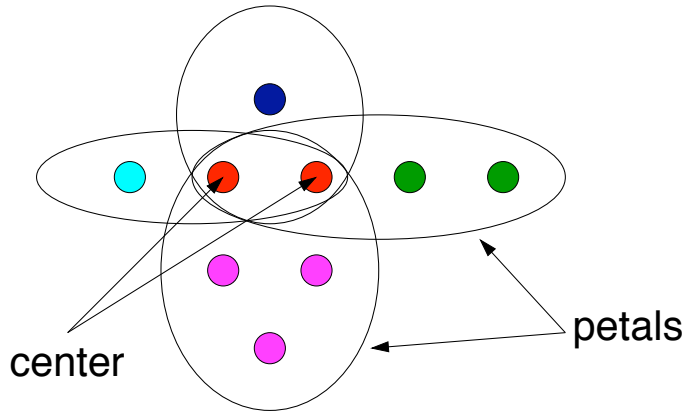
- find a matching of size $k+1$, $\Rightarrow$ YES: we can save k colors!
- find a crown decomposition, $\Rightarrow$ Reduce!
- or conclude that the graph has at most $3k$ vertices.
  $\Rightarrow 3k$ vertex kernel!

This gives a 3k vertex kernel for Dual of Vertex Coloring.

Sunflower Lemma

Sunflower lemma

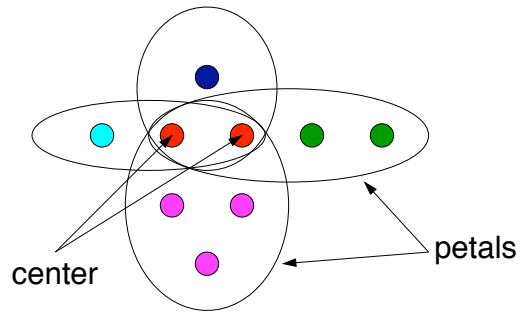
Definition: Sets S_1, S_2, ..., S_k form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \cdots \cap S_k)$ are disjoint.



Lemma: [Erdős and Rado, 1960] If the size of a set system is greater than $(p-1)^d \cdot d!$ and it contains only sets of size at most d, then the system contains a sunflower with p petals. Furthermore, in this case such a sunflower can be found in polynomial time.

Sunflowers and d-Hitting Set

d-Hitting Set: Given a collection $\mathcal{S}$ of sets of size at most d and an integer k, find a set S of k elements that intersects every set of $\mathcal{S}$.



Reduction Rule: If $k+1$ sets form a sunflower, then remove these sets from $\mathcal{S}$ and add the center C to $\mathcal{S}$ (S does not hit one of the petals, thus it has to hit the center).

Note: if the center is empty (the sets are disjoint), then there is no solution.
If the rule cannot be applied, then there are at most $O(k^d)$ sets.

Sunflowers and d-Hitting Set

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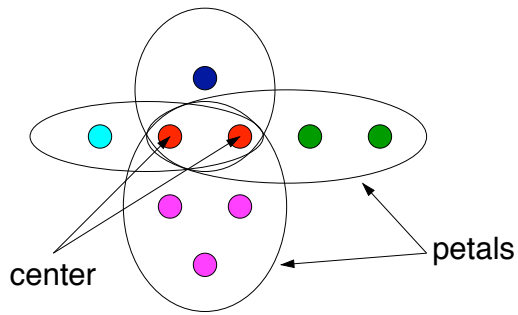
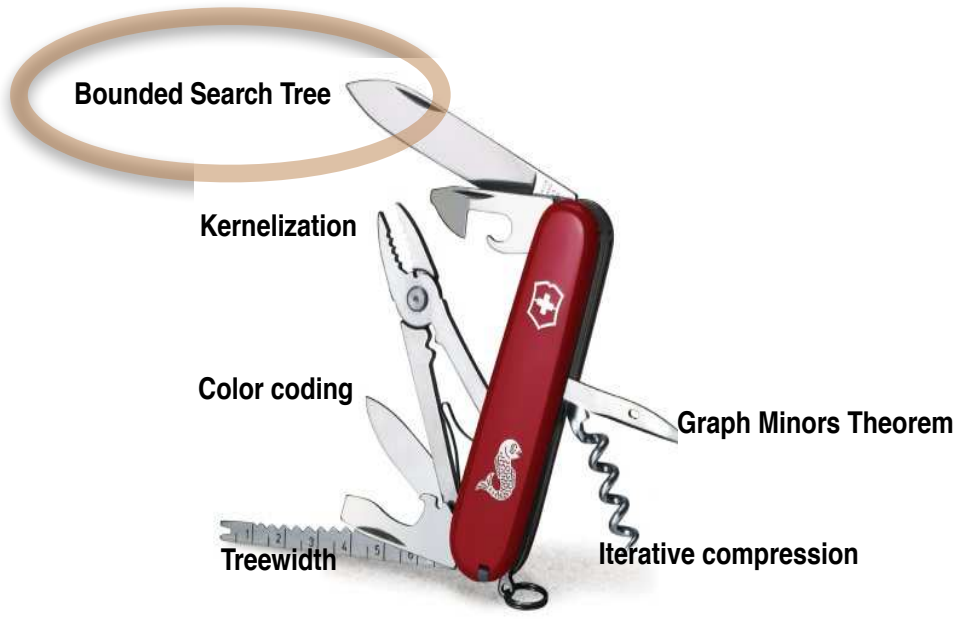


Reduction Rule (variant): Suppose more than $k+1$ sets form a sunflower.

- If the sets are disjoint $\Rightarrow$ No solution.
- Otherwise, keep only $k+1$ of the sets.

If the rule cannot be applied, then there are at most $O(k^d)$ sets.

Branching and bounded search trees



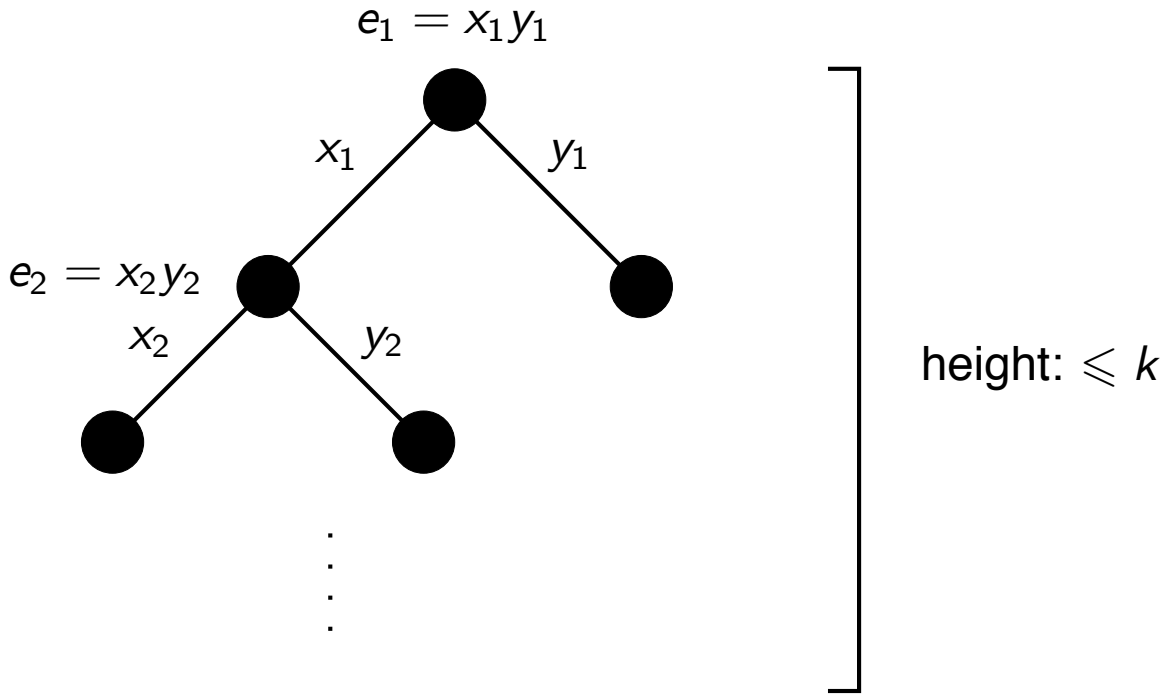
Part I: Branching Vectors

Part II: Forbidden Subgraphs

Part III: Hereditary Properties

Bounded search tree method

Recall how we solved Minimum Vertex Cover:



Bounded search tree method

We solve the problem by one or more **branching rules.**

Each rule makes a "guess" in such a way that at least one guess will lead to a correct solution.

If we have branching rules such that

- each rule branches into at most $b(k)$ directions, and
- applying a rule decreases the parameter,

then the problem can be solved in time $O^*(b(k)^k)$.

In many cases, this crude upper bound can be improved by better analysis.

Vertex Cover

Improved algorithm for Vertex Cover.

- If every vertex has degree ≤ 2, then the problem can be solved in polynomial time.
- **Branching rule:** If there is a vertex v with at least 3 neighbors, then
  - either v is in the solution,
  - or every neighbor of v is in the solution.

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

Vertex Cover

Improved algorithm for Vertex Cover.

- If every vertex has degree ≤ 2, then the problem can be solved in polynomial time.
- **Branching rule:** If there is a vertex v with at least 3 neighbors, then
  - either v is in the solution, $\Rightarrow k$ decreases by 1
  - or every neighbor of v is in the solution. $\Rightarrow k$ decreases by at least 3

Crude upper bound: $O^*(2^k)$, since the branching rule decreases the parameter.

But it is somewhat better than that, since in the second branch, the parameter decreases by at least 3.

Better analysis

Let $t(k)$ be the maximum number of leaves of the search tree if the parameter is at most k (let $t(k) = 1$ for $k \leq 0$).

$$t(k) \leq t(k-1) + t(k-3)$$

There is a standard technique for bounding such functions asymptotically.

We prove by induction that $t(k) \leq c^k$ for some $c > 1$ as small as possible.

What values of c are good? We need:

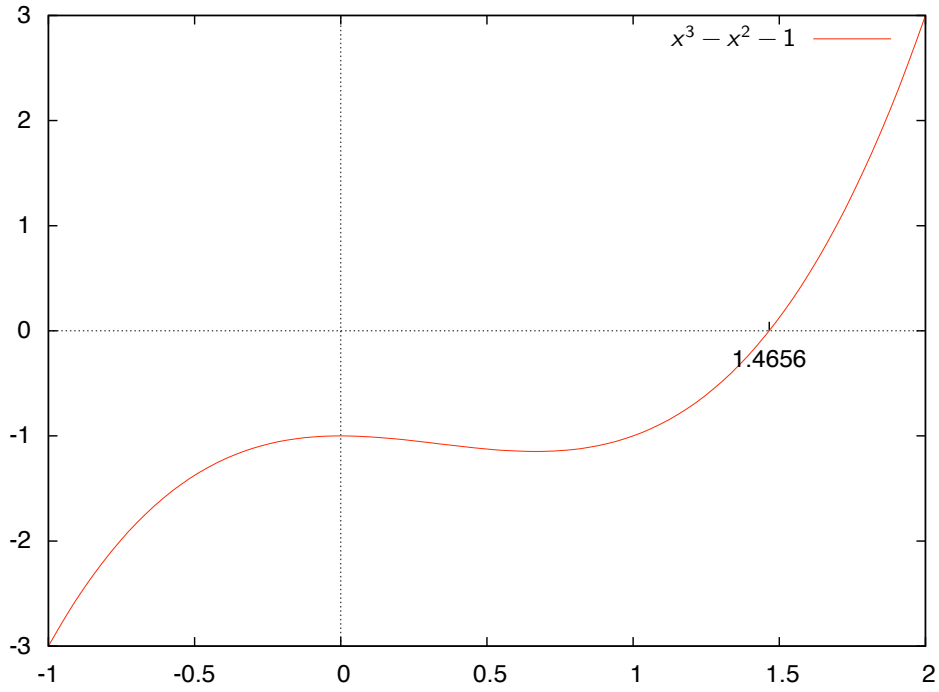
$$c^k \geq c^{k-1} + c^{k-3}$$

$$c^3 - c^2 - 1 \geq 0$$

We need to find the roots of the **characteristic equation** $c^3 - c^2 - 1 = 0$.

Note: it is always true that such an equation has a unique positive root.

Better analysis



$c = 1.4656$ is a good value $\Rightarrow t(k) \leq 1.4656^k$

$\Rightarrow$ We have a $O^*(1.4656^k)$ algorithm for Vertex Cover.

Better analysis

We showed that if $t(k) \leq t(k-1) + t(k-3)$, then $t(k) \leq 1.4656^k$ holds.

Is this bound tight? There are two questions:

- Can the function $t(k)$ be that large?
  Yes (ignoring rounding problems).
- Can the search tree of the Vertex Cover algorithm be that large?
  Difficult question, hard to answer in general.

Branching vectors

The **branching vector** of our $O^*(1.4656^k)$ Vertex Cover algoritm was $(1, 3)$.

Example: Let us bound the search tree for the branching vector $(2, 5, 6, 6, 7, 7)$.
(2 out of the 6 branches decrease the parameter by 7, etc.).

The value $c > 1$ has to satisfy:

$$c^k \geq c^{k-2} + c^{k-5} + 2c^{k-6} + 2c^{k-7}$$

$$c^7 - c^5 - c^2 - 2c - 2 \geq 0$$

Unique positive root of the characteristic equation: $1.4483 \Rightarrow$ $t(k) \leq 1.4483^k$.

It is hard to compare branching vectors intuitively.

Branching vectors

Example: The roots for branching vector (i, j) $(1 \leq i, j \leq 6)$.

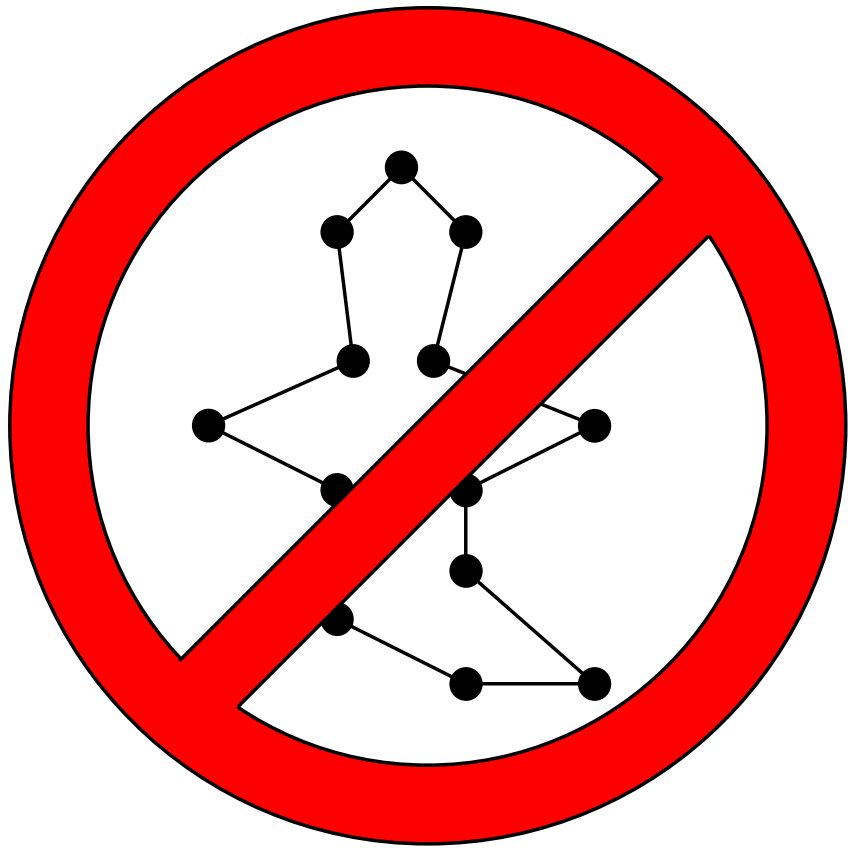
$$t(k) \leq t(k-i) + t(k-j) \Rightarrow c^k \geq c^{k-i} + c^{k-j}$$

$$c^j - c^{j-i} - 1 \geq 0$$

We compute the unique positive root.

	1	2	3	4	5	6
1	2.0000	1.6181	1.4656	1.3803	1.3248	1.2852
2	1.6181	1.4143	1.3248	1.2721	1.2366	1.2107
3	1.4656	1.3248	1.2560	1.2208	1.1939	1.1740
4	1.3803	1.2721	1.2208	1.1893	1.1674	1.1510
5	1.3248	1.2366	1.1939	1.1674	1.1487	1.1348
6	1.2852	1.2107	1.1740	1.1510	1.1348	1.1225

Forbidden subgraphs



Forbidden subgraphs

General problem class: Given a graph G and an integer k, transform G with at most k modifications (add/remove vertices/edges) into a graph having property $\mathcal{P}$.

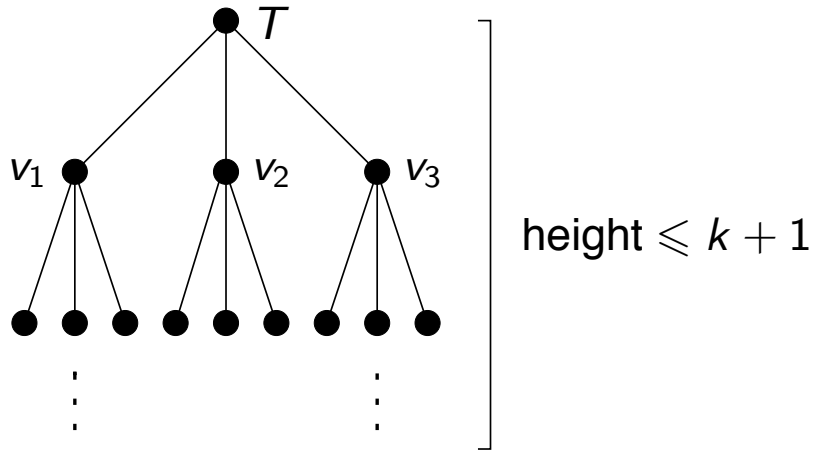
Example:
Triangle deletion: make the graph triangle-free by deleting at most k vertices.

Branching algorithm:

- If the graph is triangle-free, then we are done.
- **Branching rule:** If there is a triangle $v_1v_2v_3$, then at least one of v_1, v_2, v_3 has to be deleted $\Rightarrow$ We branch into 3 directions.

Triangle deletion

Search tree:



The search tree has at most 3^k leaves and the work to be done is polynomial at each step $\Rightarrow$ $O^*(3^k)$ time algorithm.

Note: If the answer is "NO", then the search tree has **exactly** 3^k leaves.

Hereditary properties

Definition: A graph property $\mathcal{P}$ is **hereditary** if for every $G \in \mathcal{P}$ and induced subgraph G' of G, we have $G' \in \mathcal{P}$ as well.

Examples: triangle-free, bipartite, interval graph, planar

Observation: Every hereditary property $\mathcal{P}$ can be characterized by a (finite or infinite) set $\mathcal{F}$ of forbidden induced subgraphs:

$$G \in \mathcal{P} \iff \forall H \in \mathcal{F}, H \not\subseteq_{\mathsf{ind}} G$$

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Theorem: If $\mathcal{P}$ is hereditary and can be characterized by a **finite** set $\mathcal{F}$ of forbidden induced subgraphs, then the graph modification problems corresponding to $\mathcal{P}$ are FPT.

Hereditary properties

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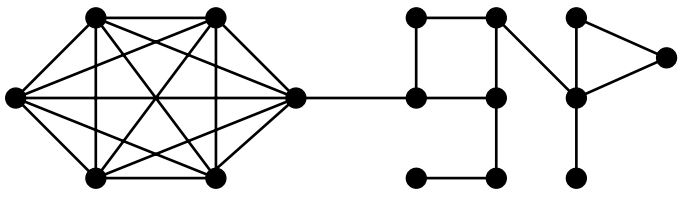
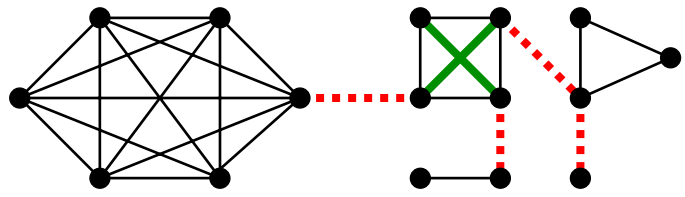
Proof:

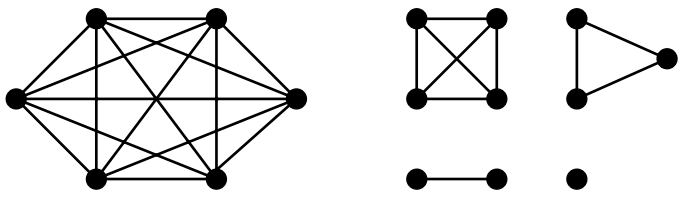
- Suppose that every graph in $\mathcal{F}$ has at most r vertices. Using brute force, we can find in time $O(n^r)$ a forbidden subgraph (if exists).
- If a forbidden subgraph exists, then we have to delete one of the at most r vertices or add/delete one of the at most $\binom{r}{2}$ edges
  $\Rightarrow$ Branching factor is a constant c depending on $\mathcal{F}$.
- The search tree has at most c^k leaves and the work to be done at each node is $O(n^r)$.

Cluster Editing

Task: Given a graph G and an integer k, add/remove at most k edges such that every component is a clique in the resulting graph.







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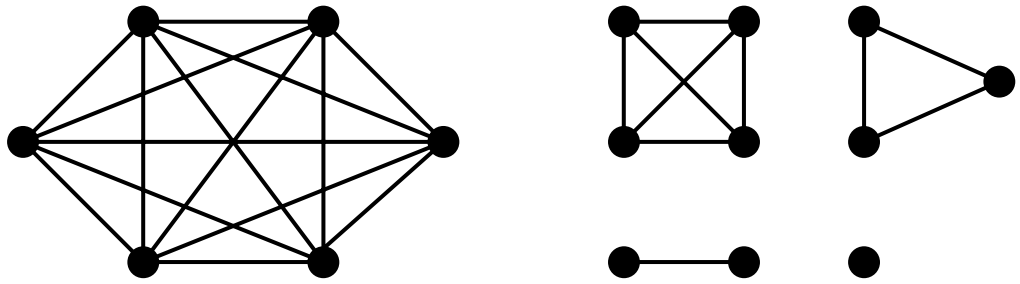
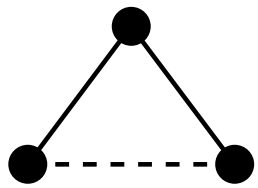


Property $\mathcal{P}$: every component is a clique.

Forbidden induced subgraph:



$O^*(3^k)$ time algorithm.

Chordal Completion

Definition: A graph is **chordal** if it does not contain an induced cycle of length greater than 3.

Chordal Completion: Given a graph G and an integer k, add at most k edges to G to make it a chordal graph.

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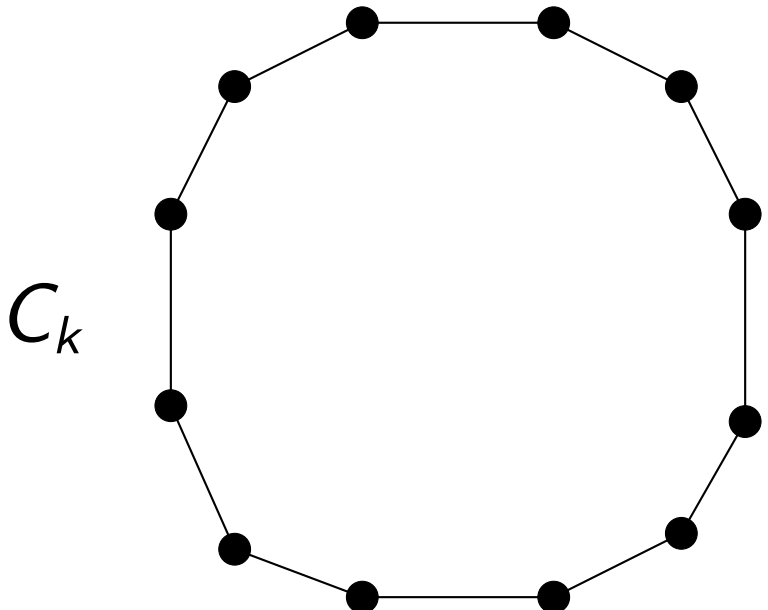
Lemma: At least $k-3$ edges are needed to make a k-cycle chordal.

Proof: By induction. $k=3$ is trivial.



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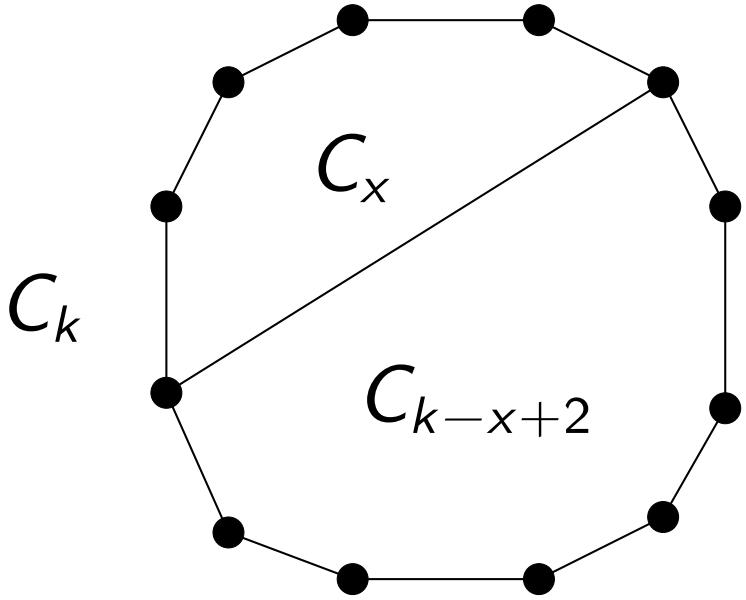


C_x: $x-3$ edges
C_{k-x+2}: $k-x-1$ edges
C_k: $(x-3)+(k-x-1)+1 = k-3$ edges

Chordal Completion

Algorithm:

- Find an induced cycle C of length at least 4 (can be done in polynomial time).
- If no such cycle exists $\Rightarrow$ Done!
- If C has more than $k+3$ vertices $\Rightarrow$ No solution!
- Otherwise, one of the

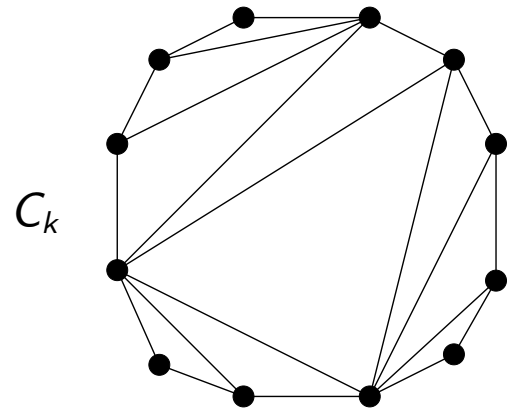
$$\binom{|C|}{2} - |C| \leq (k+3)(k+2)/2 - k = O(k^2)$$

  missing edges has to be added $\Rightarrow$ Branch!

Size of the search tree is $k^{O(k)}$.

Chordal Completion – more efficiently

Definition: Triangulation of a cycle.



Lemma: Every chordal supergraph of a cycle C contains a triangulation of the cycle C.

Lemma: The number of ways a cycle of length k can be triangulated is exactly the $(k-2)$-nd Catalan number

$$C_{k-2} = \frac{1}{k-1}\binom{2(k-2)}{k-2} \leq 4^{k-3}.$$

Chordal Completion – more efficiently

Algorithm:

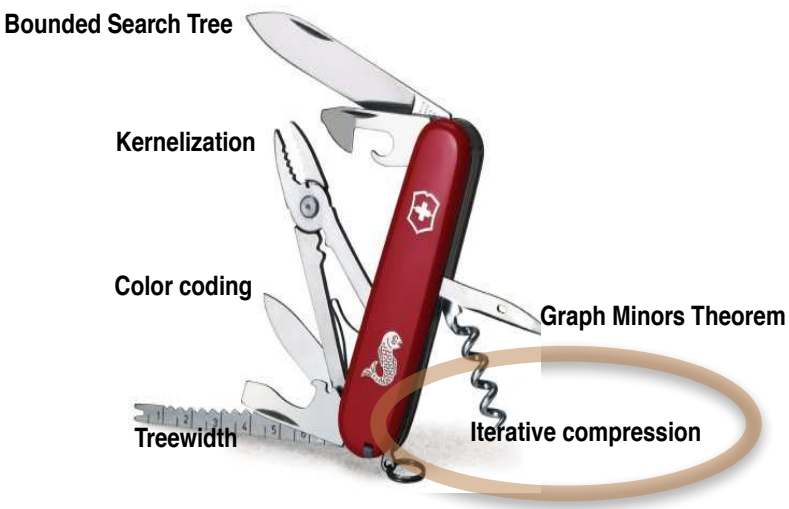
- Find an induced cycle C of length at least 4 (can be done in polynomial time).
- If no such cycle exists $\Rightarrow$ Done!
- If C has more than $k+3$ vertices $\Rightarrow$ No solution!
- Otherwise, one of the $\leq 4^{|C|-3}$ triangulations has to be in the solution $\Rightarrow$ Branch!

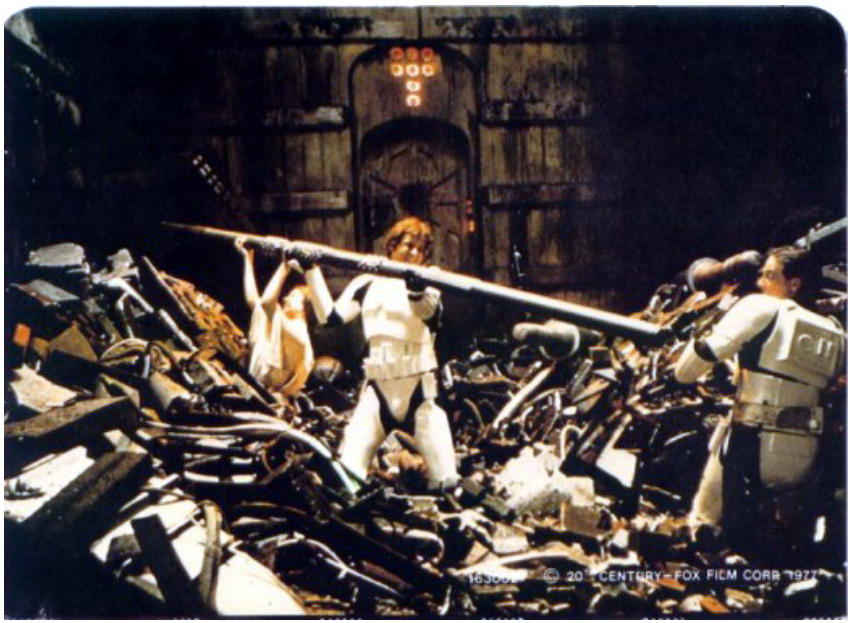
Claim: Search tree has at most $T_k = 4^k$ leaves.

Proof: By induction. Number of leaves is at most

$$T_k \leq 4^{|C|-3} \cdot T_{k-(|C|-3)} \leq 4^{|C|-3} \cdot 4^{k-(|C|-3)} = 4^k.$$

Iterative compression





Iterative compression

- A surprising small, but very powerful trick.
- Most useful for deletion problems: delete k things to achieve some property.
- Demonstration: Odd Cycle Transversal aka Bipartite Deletion aka Graph Bipartization: Given a graph G and an integer k, delete k vertices to make the graph bipartite.
- Forbidden induced subgraphs: odd cycles. There is no bound on the size of odd cycles.

Bipartite Deletion

Solution based on iterative compression:

- **Step 1:**
  Solve the **annotated problem** for bipartite graphs:

  Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.

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- **Step 2:**
  Solve the **compression problem** for general graphs:

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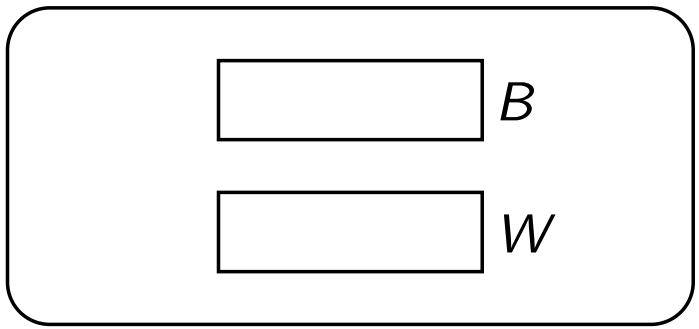
- **Step 3:**
  Apply the magic of iterative compression...

Step 1: The annotated problem

Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.



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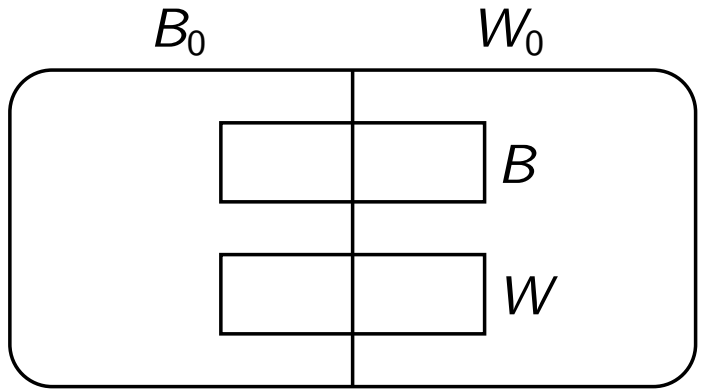

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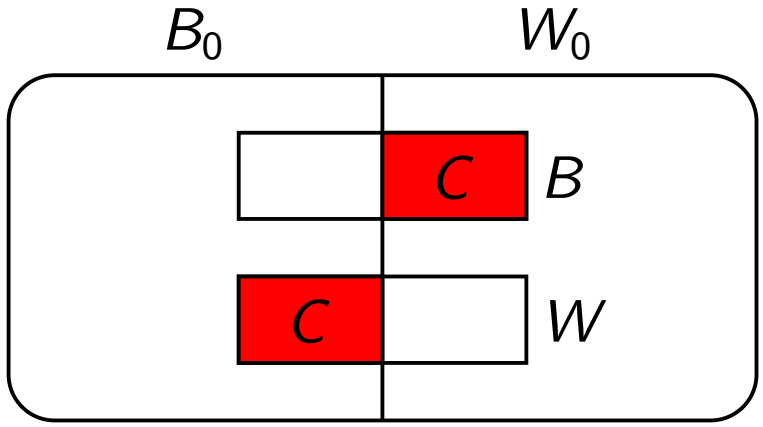

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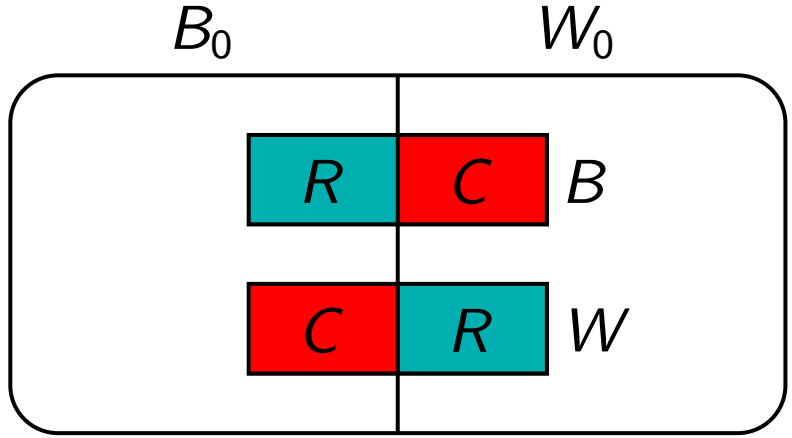

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Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

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Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Proof:

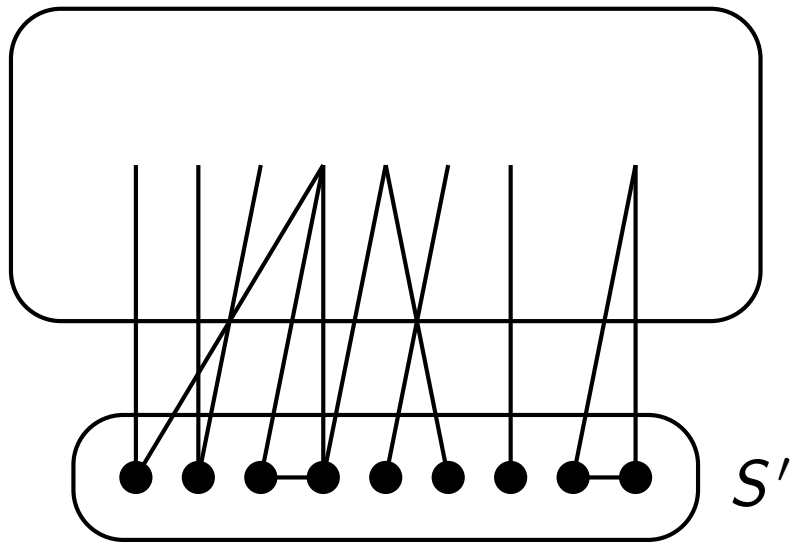
$\Rightarrow$ In a 2-coloring of $G \setminus S$, each vertex either remained the same color or changed color. Adjacent vertices do the same, thus every component either changed or remained.

$\Leftarrow$ Flip the coloring of those components of $G \setminus S$ that contain vertices from $C \setminus S$. No vertex of R is flipped.

Algorithm: Using max-flow min-cut techniques, we can check if there is a set S that separates C and R. It can be done in time $O(k|E(G)|)$ using k iterations of the Ford-Fulkerson algorithm.

Step 2: The compression problem

Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.



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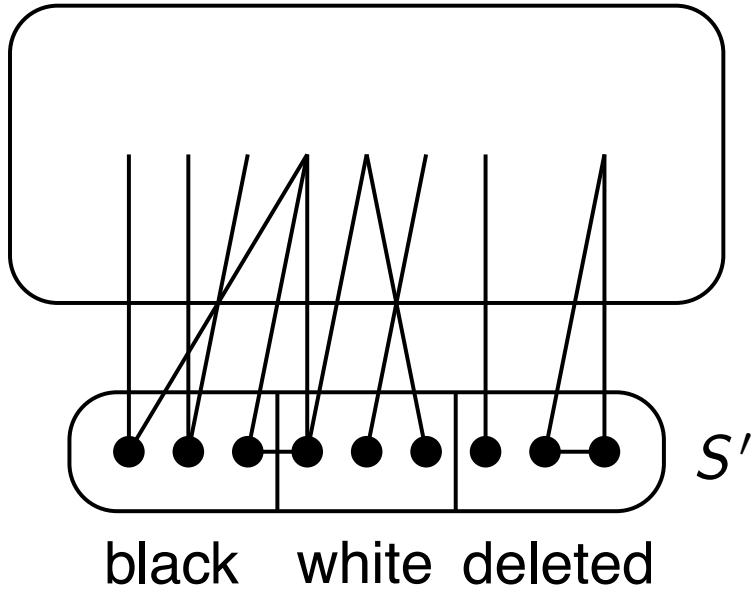

Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.

Trivial check: no edge between two black or two white vertices.

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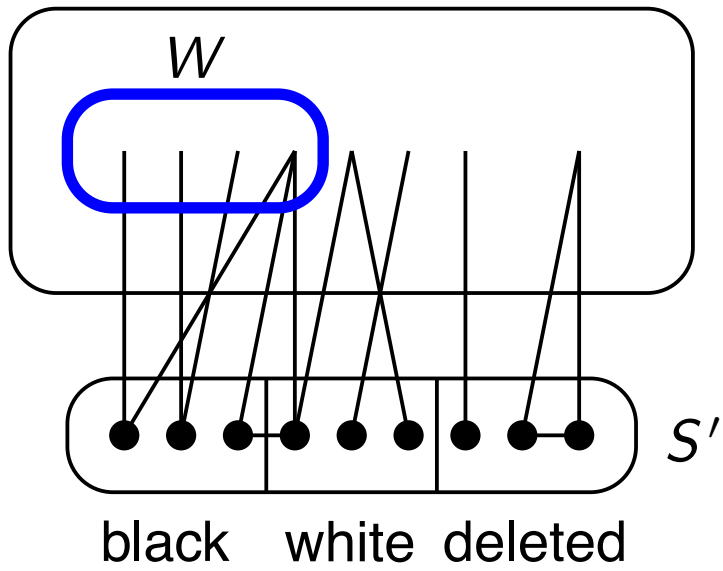

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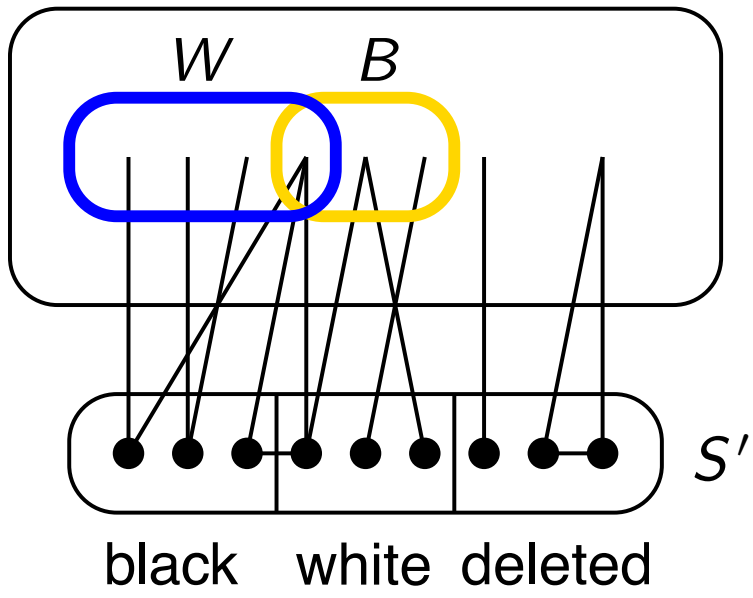

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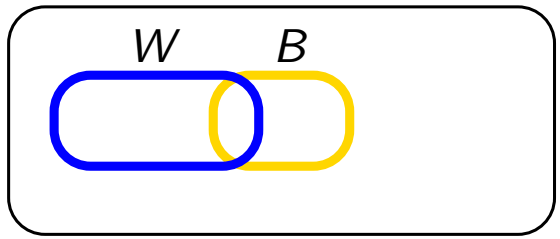

The vertices of S' can be disregarded. Thus we need to solve the annotated problem on the bipartite graph $G \setminus S'$.

Running time: $O(3^k \cdot k|E(G)|)$ time.

Step 3: Iterative compression

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Let $V(G) = \{v_1, \ldots, v_n\}$ and let G_i be the graph induced by $\{v_1, \ldots, v_i\}$.

For every i, we find a set S_i of size k such that $G_i \setminus S_i$ is bipartite.

- For G_k, the set $S_k = \{v_1, \ldots, v_k\}$ is a trivial solution.
- If S_{i-1} is known, then $S_{i-1} \cup \{v_i\}$ is a set of size $k+1$ whose deletion makes G_i bipartite $\Rightarrow$ We can use the compression algorithm to find a suitable S_i in time $O(3^k \cdot k|E(G_i)|)$.

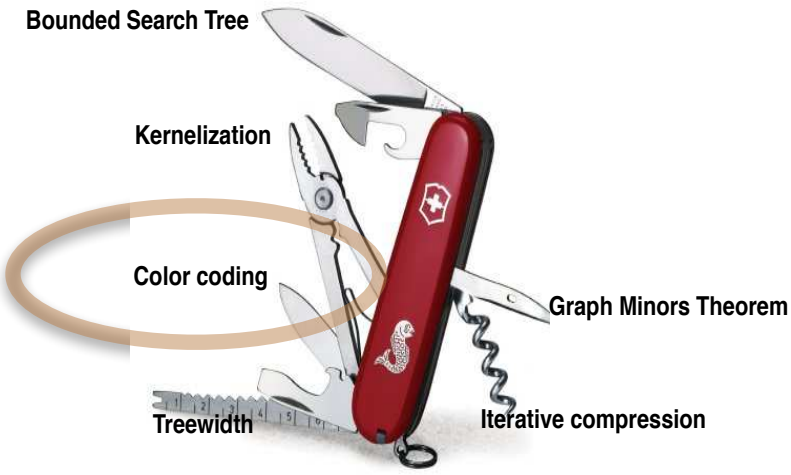
Step 3: Iterative Compression

Bipartite-Deletion(G, k)

1. $S_k = \{v_1, \ldots, v_k\}$
2. for $i := k + 1$ to n
3. Invariant: $G_{i-1} \setminus S_{i-1}$ is bipartite.
4. Call Compression$(G_i, S_{i-1} \cup \{v_i\})$
5. If the answer is "NO" $\Rightarrow$ return "NO"
6. If the answer is a set $X \Rightarrow S_i := X$
7. Return the set S_n

Running time: the compression algorithm is called n times and everything else can be done in linear time
$\Rightarrow O(3^k \cdot k|V(G)| \cdot |E(G)|)$ time algorithm.

Color coding



Color coding

- Works best when we need to ensure that a small number of "things" are disjoint.
- We demonstrate it on the problem of finding an s-t path of length **exactly** k.
- Randomized algorithm, but can be derandomized using a standard technique.
- Very robust technique, we can use it as an "opening step" when investigating a new problem.

k-Path

Task: Given a graph G, an integer k, two vertices s, t, find a **simple** s-t path with exactly k internal vertices.

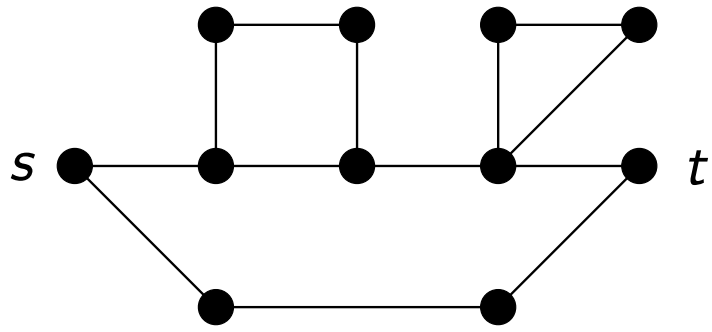
Note: Finding such a **walk** can be done easily in polynomial time.

Note: The problem is clearly NP-hard, as it contains the s-t Hamiltonian Path problem.

The k-Path algorithm can be used to check if there is a cycle of length exactly k in the graph.

k-Path

- Assign colors from $[k]$ to vertices $V(G) \setminus \{s, t\}$ uniformly and independently at random.



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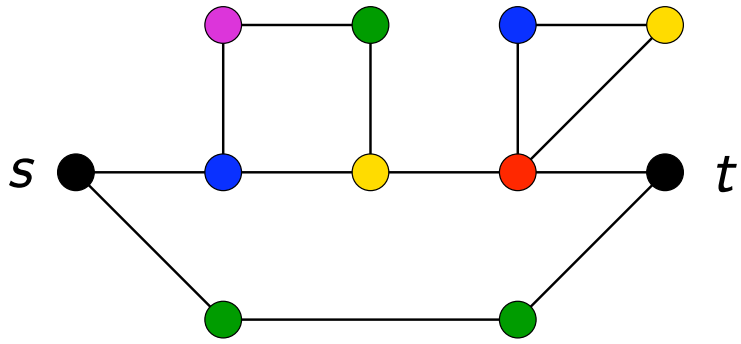

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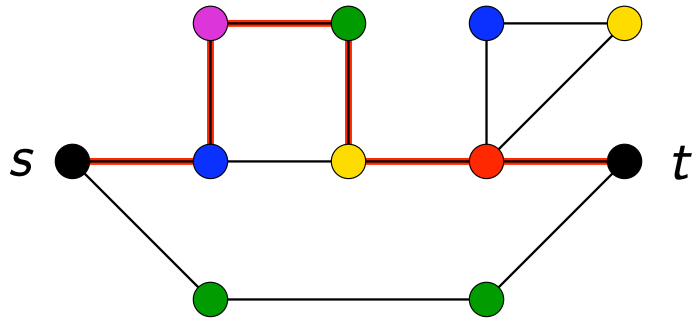

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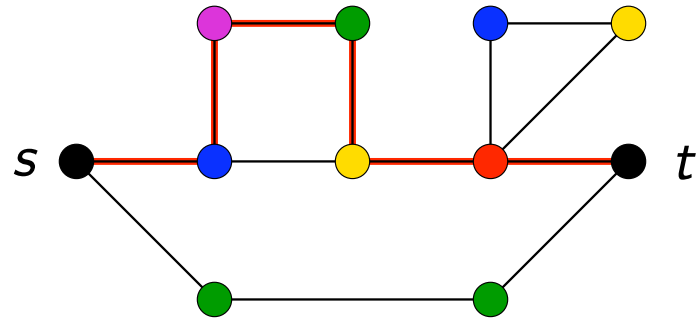

- Check if there is a **colorful** s-t path: a path where each color appears exactly once on the internal vertices; output "YES" or "NO".
  - If there is no s-t k-path: no such colorful path exists $\Rightarrow$ "NO".
  - If there is an s-t k-path: the probability that such a path is colorful is

$$\frac{k!}{k^k} > \frac{(\frac{k}{e})^k}{k^k} = e^{-k},$$

    thus the algorithm outputs "YES" with at least that probability.

Error probability

- **Useful fact:** If the probability of success is at least p, then the probability that the algorithm **does not** say "YES" after $1/p$ repetitions is at most

$$(1-p)^{1/p} < \left(e^{-p}\right)^{1/p} = 1/e \approx 0.38$$

- Thus if $p > e^{-k}$, then error probability is at most $1/e$ after e^k repetitions.
- Repeating the whole algorithm a constant number of times can make the error probability an arbitrary small constant.
- For example, by trying $100 \cdot e^k$ random colorings, the probability of a wrong answer is at most $1/e^{100}$.

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It remains to see how a colorful s-t path can be found.

Method 1: Trying all permutations.
Method 2: Dynamic programming.

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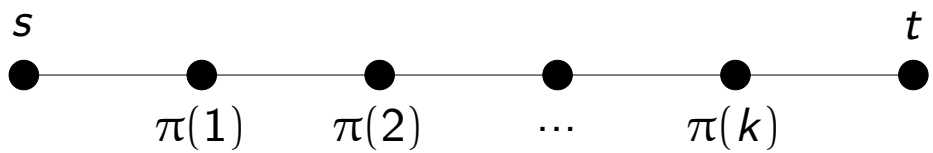
The colors encountered on a colorful s-t path form a permutation π of $\{1, 2, \ldots, k\}$:



We try all possible $k!$ permutations. For a fixed π, it is easy to check if there is a path with this order of colors.

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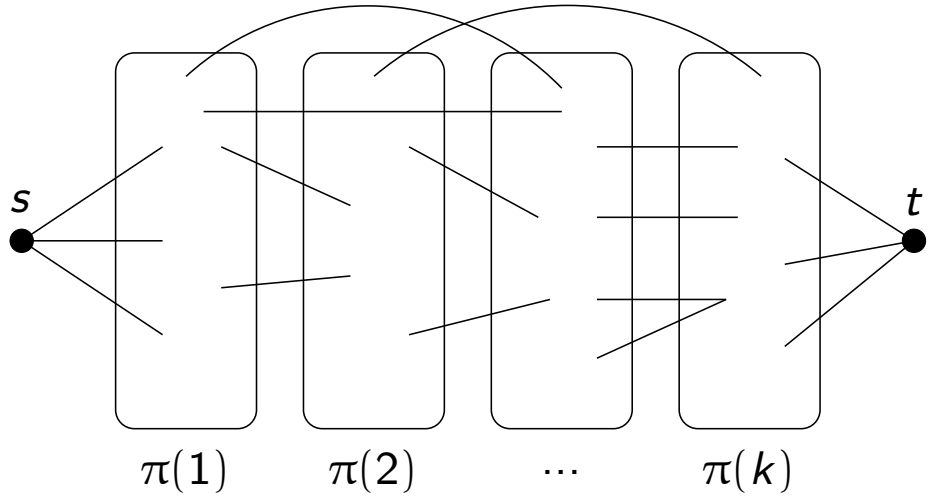

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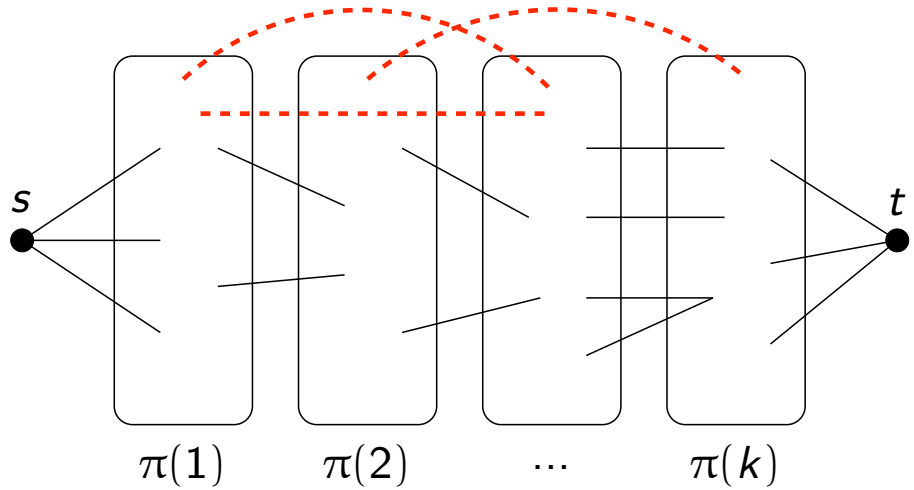

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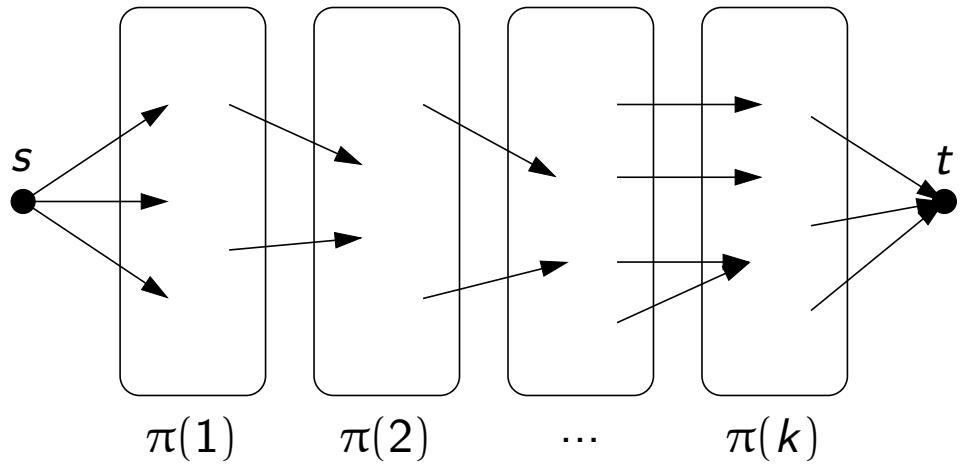

- Edges connecting nonadjacent color classes are removed.
- The remaining edges are directed.
- All we need to check if there is a directed s-t path.
- Running time is $O(k! \cdot |E(G)|)$.

Method 2: Dynamic Programming

We introduce $2^k \cdot |V(G)|$ Boolean variables:

$x(v, C) =$ TRUE for some $v \in V(G)$ and $C \subseteq [k]$

$\Updownarrow$

There is an s-v path where each color in C appears exactly once and no other color appears.

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If we know every $x(v, C)$ with $|C| = i$, then we can determine every $x(v, C)$ with $|C| = i + 1 \Rightarrow$ All the values can be determined in time $O(2^k \cdot |E(G)|)$.

There is a colorful s-t path $\iff$ $x(v, [k]) = \text{TRUE}$ for some neighbor of t.

Derandomization

Using Method 2, we obtain a $O^*((2e)^k)$ time algorithm with constant error probability. How to make it deterministic?

Definition: A family $\mathcal{H}$ of functions $[n] \to [k]$ is a k-**perfect** family of hash functions if for every $S \subseteq [n]$ with $|S| = k$, there is a $h \in \mathcal{H}$ such that $h(x) \neq h(y)$ for any $x, y \in S$, $x \neq y$.

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Instead of trying $O(e^k)$ random colorings, we go through a k-perfect family $\mathcal{H}$ of functions $V(G) \to [k]$. If there is a solution $\Rightarrow$ The internal vertices S are colorful for at least one $h \in \mathcal{H} \Rightarrow$ Algorithm outputs "YES".

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Theorem: There is a k-perfect family of functions $[n] \to [k]$ having size $2^{O(k)} \log n$
(and can be constructed in time polynomial in the size of the family).

$\Rightarrow$ There is a **deterministic** $2^{O(k)} \cdot n^{O(1)}$ time algorithm for the k-Path problem.

k-Disjoint Triangles

Task: Given a graph G and an integer k, find k vertex disjoint triangles.

Step 1: Choose a random coloring $V(G) \to [3k]$.

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Step 2: Check if there is a colorful solution, where the $3k$ vertices of the k triangles use distinct colors.

- **Method 1:** Try every permutation π of $[3k]$ and check if there are triangles with colors $(\pi(1), \pi(2), \pi(3))$, $(\pi(4), \pi(5), \pi(6))$, $\ldots$
- **Method 2:** Dynamic programming. For $C \subseteq [3k]$ and $|C| = 3i$, let $x(C) =$ TRUE if and only if there are $|C|/3$ disjoint triangles using exactly the colors in C.

$$x(C) = \bigvee_{\{c_1,c_2,c_3\} \subseteq C} (x(C \setminus \{c_1, c_2, c_3\}) \wedge \exists \triangle \text{ with colors } c_1, c_2, c_3)$$

k-Disjoint Triangles

Step 3: Colorful solution exists with probability at least e^{-3k}, which is a lower bound on the probability of a correct answer.

Running time: constant error probability after e^{3k} repetitions $\Rightarrow$ running time is $O^*((2e)^{3k})$ (using Method 2).

Derandomization: $3k$-perfect family of functions instead of random coloring. Running time is $2^{O(k)} \cdot n^{O(1)}$.

Color coding

We have seen that color coding can be used to find paths, cycles of length k, or a set of k disjoint triangles.

What other structures can be found efficiently with this technique?

The key is treewidth:

Theorem: Given two graph H, G, it can be decided if H is a subgraph of G in time $2^{O(|V(H)|)} \cdot |V(G)|^{O(w)}$, where w is the treewidth of G.

Thus if H belongs to a class of graphs with bounded treewidth, then the subgraph problem is FPT.