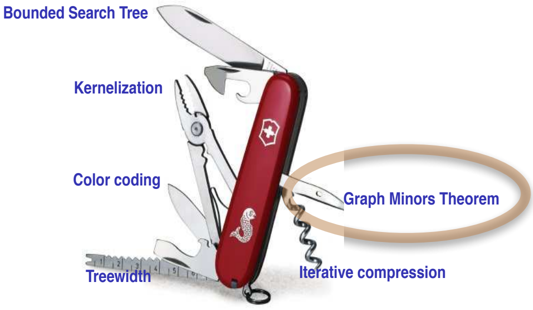
FEDOR V. FOMIN

Parameterized Algorithms II

PART I

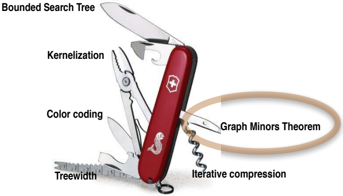
St. Petersburg, 2011

Graph Minors



Graph Minors

Neil Robertson



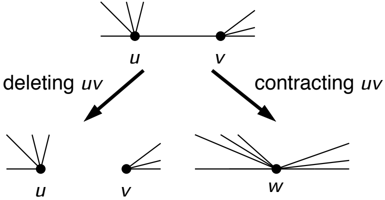
Paul Seymour

Graph Minors

- Some consequences of the Graph Minors Theorem give a quick way of showing that certain problems are FPT.
- However, the function $f(k)$ in the resulting FPT algorithms can be HUGE, completely impractical.
- History: motivation for FPT.
- Parts and ingredients of the theory are useful for algorithm design.
- New algorithmic results are still being developed.

Graph Minors

Definition: Graph H is a **minor** G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

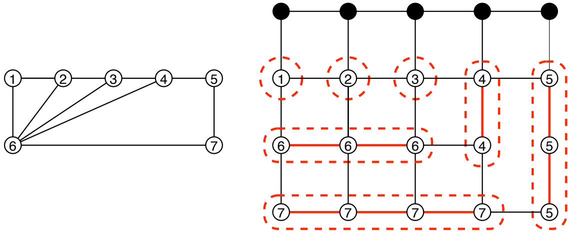


Example: A triangle is a minor of a graph G if and only if G has a cycle (i.e., it is not a forest).

Graph minors

Equivalent definition: Graph H is a **minor** of G if there is a mapping ϕ that maps each vertex of H to a connected subset of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.



Minor closed properties

Definition: A set $\mathcal{G}$ of graphs is **minor closed** if whenever $G \in \mathcal{G}$ and $H \le G$, then $H \in \mathcal{G}$ as well.

Examples of minor closed properties:

planar graphs
acyclic graphs (forests)
graphs having no cycle longer than k
empty graphs

Examples of not minor closed properties:

complete graphs
regular graphs
bipartite graphs

Forbidden minors

Let $\mathcal{G}$ be a minor closed set and let $\mathcal{F}$ be the set of "minimal bad graphs": $H \in \mathcal{F}$ if $H \notin \mathcal{G}$, but every proper minor of H is in $\mathcal{G}$.

Characterization by forbidden minors:

$$G \in \mathcal{G} \Longleftrightarrow \forall H \in \mathcal{F}, H \not\preceq G$$

The set $\mathcal{F}$ is the **obstruction set** of property $\mathcal{G}$.

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The set $\mathcal{F}$ is the **obstruction set** of property $\mathcal{G}$.

Theorem: [Wagner] A graph is planar if and only if it does not have a K_5 or $K_{3,3}$ minor.

In other words: the obstruction set of planarity is $\mathcal{F} = \{K_5, K_{3,3}\}$.

Does every minor closed property have such a finite characterization?

Graph Minors Theorem

Theorem: [Robertson and Seymour] Every minor closed property $\mathcal{G}$ has a finite obstruction set.

Note: The proof is contained in the paper series "Graph Minors I–XX".

Note: The size of the obstruction set can be astronomical even for simple properties.

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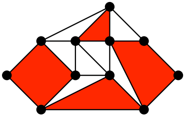
Note: The size of the obstruction set can be astronomical even for simple properties.

Theorem: [Robertson and Seymour] For every fixed graph H, there is an $O(n^3)$ time algorithm for testing whether H is a minor of the given graph G.

Corollary: For every minor closed property $\mathcal{G}$, there is an $O(n^3)$ time
algorithm for testing whether a given graph G is in $\mathcal{G}$.

Applications

PLANAR FACE COVER: Given a graph G and an integer k, find an embedding of planar graph G such that there are k faces that cover all the vertices.



One line argument:

For every fixed k, the class $\mathcal{G}_k$ of graphs of yes-instances is minor closed.

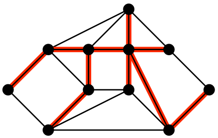
$$\Downarrow$$

For every fixed k, there is a $O(n^3)$ time algorithm for PLANAR FACE COVER.

Note: non-uniform FPT.

Applications

k-Leaf Spanning Tree: Given a graph G and an integer k, find a spanning tree with **at least** k leaves.



Technical modification: Is there such a spanning tree for at least one component of G?

One line argument:

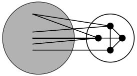
For every fixed k, the class $\mathcal{G}_k$ of no-instances is minor closed.

$$\Downarrow$$

For every fixed k, k-Leaf Spanning Tree can be solved in time $O(n^3)$.

$\mathcal{G} + k$ vertices

Let $\mathcal{G}$ be a graph property, and let $\mathcal{G} + kv$ contain graph G if there is a set $S \subseteq V(G)$ of k vertices such that $G \setminus S \in \mathcal{G}$.



Lemma: If $\mathcal{G}$ is minor closed, then $\mathcal{G} + kv$ is minor closed for every fixed k.

$\Rightarrow$ It is (nonuniform) FPT to decide if G can be transformed into a member of $\mathcal{G}$ by deleting k vertices.

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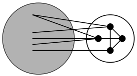

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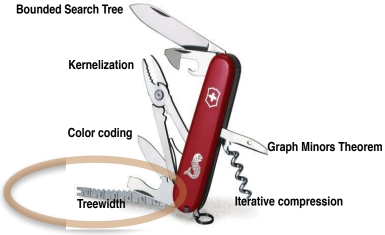
- If $\mathcal{G}$ = forests $\Rightarrow$ $\mathcal{G} + kv$ = graphs that can be made acyclic by the deletion of k vertices $\Rightarrow$ Feedback Vertex Set is FPT.
- If $\mathcal{G}$ = planar graphs $\Rightarrow$ $\mathcal{G} + kv$ = graphs that can be made planar by the deletion of k vertices (k-apex graphs) $\Rightarrow$ k-Apex Graph is FPT.
- If $\mathcal{G}$ = empty graphs $\Rightarrow$ $\mathcal{G} + kv$ = graphs with vertex cover number at most k $\Rightarrow$ Vertex Cover is FPT.

Treewidth


Bounded Search Tree
Kernelization
Color coding
Graph Minors Theorem
Treewidth
Iterative compression


Chandelier Tree

TREEWIDTH

Introduction and definition

Part I: Algorithms for bounded treewidth graphs.

Part II: Graph-theoretic properties of treewidth.

Part III: Applications for general graphs.

The Party Problem

Party Problem

Problem:	Invite some colleagues for a party.
Maximize:	The total fun factor of the invited people.
Constraint:	Everyone should be having fun.

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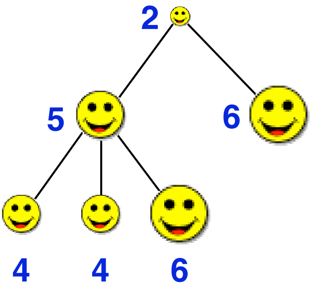

- **Input:** A tree with weights on the vertices.
- **Task:** Find an independent set of maximum weight.

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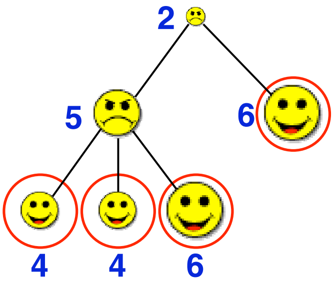

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Solving the Party Problem

Dynamic programming paradigm: We solve a large number of subproblems that depend on each other. The answer is a single subproblem.

T_v: the subtree rooted at v.
$A[v]$: max. weight of an independent set in T_v
$B[v]$: max. weight of an independent set in T_v that does not contain v

Goal: determine $A[r]$ for the root r.

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Goal: determine $A[r]$ for the root r.

Method:
Assume $v_1, \ldots, v_k$ are the children of v. Use the recurrence relations

$$B[v] = \sum_{i=1}^{k} A[v_i]$$
$$A[v] = \max\{B[v]\ ,\ w(v) + \sum_{i=1}^{k} B[v_i]\}$$

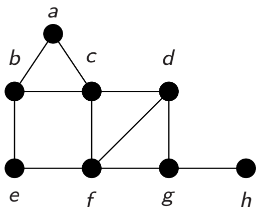
The values $A[v]$ and $B[v]$ can be calculated in a bottom-up order (the leaves are trivial).

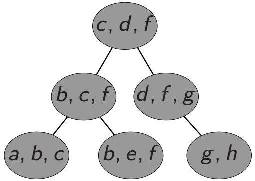
Treewidth

Treewidth: A measure of how "tree-like" the graph is.
(Introduced by Robertson and Seymour.)

Tree decomposition: Vertices are arranged in a tree structure satisfying the following properties:

1. If u and v are neighbors, then there is a bag containing both of them.
2. For every vertex v, the bags containing v form a connected subtree.





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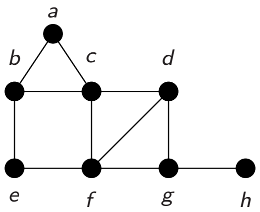

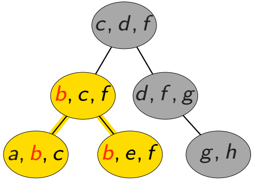

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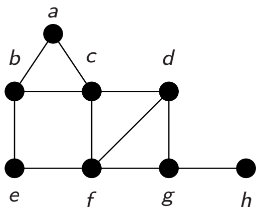

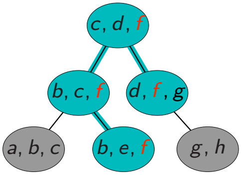

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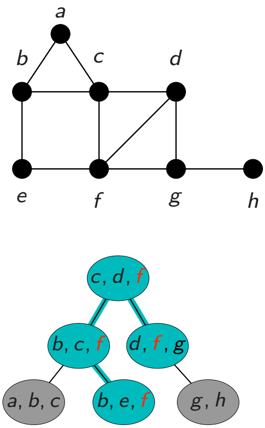
Width of the decomposition: largest bag size -1.

treewidth: width of the best decomposition.

Fact: treewidth $= 1$ iff graph is a forest



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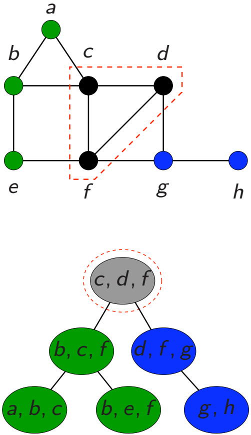

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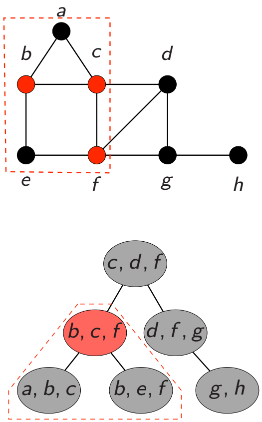

Finding tree decompositions

Fact: It is NP-hard to determine the treewidth of a graph (given a graph G and an integer w, decide if the treewidth of G is at most w), but there is a polynomial-time algorithm for every fixed w.

Finding tree decompositions

Fact: [Bodlaender's Theorem] For every fixed w, there is a linear-time algorithm that finds a tree decomposition of width w (if exists).

$\Rightarrow$ Deciding if treewidth is at most w is fixed-parameter tractable.
$\Rightarrow$ If we want an FPT algorithm parameterized by treewidth w of the input graph, then we can assume that a tree decomposition of width w is available.

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$\Rightarrow$ If we want an FPT algorithm parameterized by treewidth w of the input graph, then we can assume that a tree decomposition of width w is available.

Running time is $2^{O(w^3)} \cdot n$. Sometimes it is better to use the following results instead:

Fact: There is a $O(3^{3w} \cdot w \cdot n^2)$ time algorithm that finds a tree decomposition of width $4w + 1$, if the treewidth of the graph is at most w.

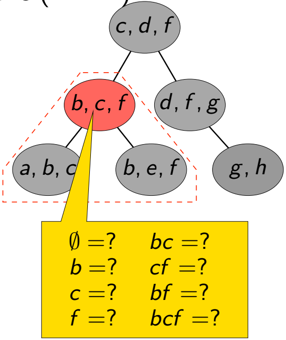
Fact: There is a polynomial-time algorithm that finds a tree decomposition of width $O(w\sqrt{\log w})$, if the treewidth of the graph is at most w.

Part I:

Algoritmhs for bounded-treewidth graphs

WEIGHTED MAX INDEPENDENT SET
and tree decompositions

Fact: Given a tree decomposition of width w, WEIGHTED MAX INDEPENDENT SET can be solved in time $O(2^w \cdot n)$.



B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

Generalizing our solution for trees:

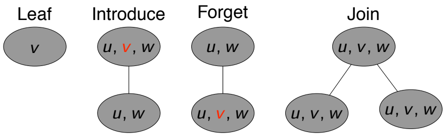
Instead of computing 2 values $A[v]$, $B[v]$ for each **vertex** of the graph, we compute $2^{|B_x|} \leq 2^{w+1}$ values for each bag B_x.

$M[x, S]$: the maximum weight of an independent set $I \subseteq V_x$ with $I \cap B_x = S$.
How to determine $M[x, S]$ if all the values are known for the children of x?

Nice tree decompositions

Definition: A rooted tree decomposition is **nice** if every node x is one of the following 4 types:

- **Leaf:** no children, $|B_x| = 1$
- **Introduce:** 1 child y, $B_x = B_y \cup \{v\}$ for some vertex v
- **Forget:** 1 child y, $B_x = B_y \setminus \{v\}$ for some vertex v
- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$

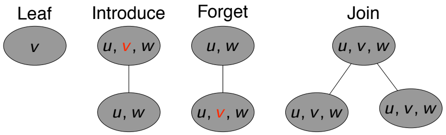


Fact: A tree decomposition of width w and n nodes can be turned into a nice tree decomposition of width w and $O(wn)$ nodes in time $O(w^2 n)$.

Weighted Max Independent Set
and nice tree decompositions

- **Leaf:** no children, $|B_x| = 1$
  Trivial!
- **Introduce:** 1 child y, $B_x = B_y \cup \{v\}$ for some vertex v

$$m[x,S] = \begin{cases} m[y,S] & \text{if } v \notin S, \\ m[y, S \setminus \{v\}] + w(v) & \text{if } v \in S \text{ but } v \text{ has no neighbor in } S, \\ -\infty & \text{if } S \text{ contains } v \text{ and its neighbor.} \end{cases}$$



Weighted Max Independent Set
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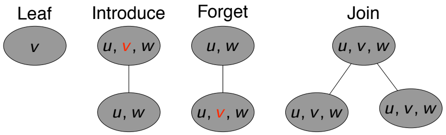
- **Forget:** 1 child y, $B_x = B_y \setminus \{v\}$ for some vertex v

$$m[x, S] = \max\{m[y, S], m[y, S \cup \{v\}]\}$$

- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$

$$m[x, S] = m[y_1, S] + m[y_2, S] - w(S)$$



WEIGHTED MAX INDEPENDENT SET and nice tree decompositions

- **Forget:** 1 child y, $B_x = B_y \setminus \{v\}$ for some vertex v

$$m[x, S] = \max\{m[y, S], m[y, S \cup \{v\}]\}$$

- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$

$$m[x, S] = m[y_1, S] + m[y_2, S] - w(S)$$

There are at most $2^{w+1} \cdot n$ subproblems $m[x, S]$ and each subproblem can be solved in constant time (assuming the children are already solved) $\Rightarrow$ Running time is $O(2^w \cdot n)$.

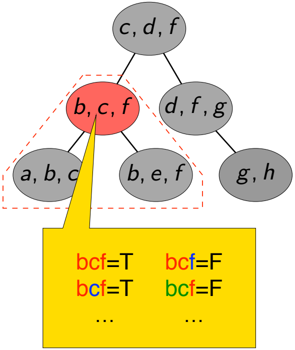
$\Rightarrow$ WEIGHTED MAX INDEPENDENT SET is FPT parameterized by treewidth $\Rightarrow$ WEIGHTED MIN VERTEX COVER is FPT parameterized by treewidth.

3-Coloring and tree decompositions

Fact: Given a tree decomposition of width w, 3-Coloring can be solved in $O(3^w \cdot n)$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

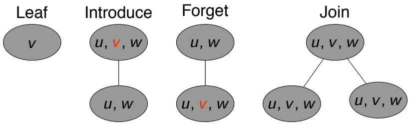
For every node x and coloring $c : B_x \to \{1, 2, 3\}$, we compute the Boolean value $E[x, c]$, which is true if and only if c can be extended to a proper 3-coloring of V_x.



How to determine $E[x, c]$ if all the values are known for the children of x?

3-Coloring and nice tree decompositions

- **Leaf:** no children, $|B_x| = 1$
  Trivial!
- **Introduce:** 1 child y, $B_x = B_y \cup \{v\}$ for some vertex v
  If $c(v) \neq c(u)$ for every neighbor u of v, then
  $E[x, c] = E[y, c']$, where c' is c restricted to B_y.
- **Forget:** 1 child y, $B_x = B_y \setminus \{v\}$ for some vertex v
  $E[x, c]$ is true if $E[y, c']$ is true for one of the 3 extensions of c to B_y.
- **Join:** 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$
  $E[x, c] = E[y_1, c] \wedge E[y_2, c]$



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There are at most $3^{w+1} \cdot n$ subproblems $E[x, c]$ and each subproblem can be solved in constant time (assuming the children are already solved).

$\Rightarrow$ Running time is $O(3^w \cdot n)$.

$\Rightarrow$ 3-Coloring is FPT parameterized by treewidth.

Vertex coloring

More generally:

Fact: Given a tree decomposition of width w, c-Coloring can be solved in $O^*(c^w)$.

Exercise: Every graph of treewidth at most w can be colored with $w + 1$ colors.

Fact: Given a tree decomposition of width w, Vertex Coloring can be solved in time $O^*(w^w)$.

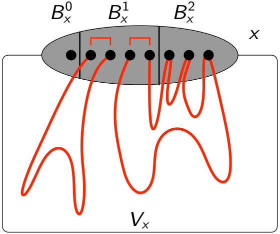
$\Rightarrow$ Vertex Coloring is FPT parameterized by treewidth.

Hamiltonian cycle and tree decompositions

Fact: Given a tree decomposition of width w, Hamiltonian cycle can be solved in time $w^{O(w)} \cdot n$.

B_x: vertices appearing in node x.
V_x: vertices appearing in the subtree rooted at x.

If H is a Hamiltonian cycle, then the subgraph $H[V_x]$ is a set of paths with endpoints in B_x.



Hamiltonian cycle and tree decompositions

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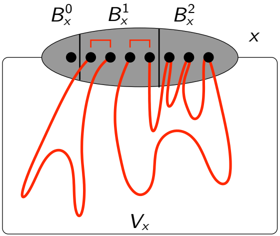
V_x: vertices appearing in the subtree rooted at x.



If H is a Hamiltonian cycle, then the subgraph $H[V_x]$ is a set of paths with endpoints in B_x.

What are the important properties of $H[V_x]$ "seen from the outside world"?

- The subsets B_x^0, B_x^1, B_x^2 of B_x having degree 0, 1, and 2.
- The matching M of B_x^1.

Number of subproblems (B_x^0, B_x^1, B_x^2, M) for each node x: at most $3^w \cdot w^w$.

Hamiltonian cycle and nice tree decompositions

For each subproblem (B_x^0, B_x^1, B_x^2, M), we have to determine if there is a set of paths with this pattern.

How to do this for the different types of nodes?
(Assuming that all the subproblems are solved for the children.)

Leaf: no children, $|B_x| = 1$

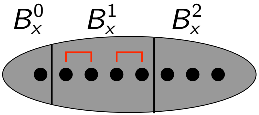
Trivial!

Hamiltonian cycle and nice tree decompositions

Solving subproblem (B_x^0, B_x^1, B_x^2, M) of node x.

Forget: 1 child y, $B_x = B_y \setminus \{v\}$ for some vertex v

In a solution H of (B_x^0, B_x^1, B_x^2, M), vertex v has degree 2. Thus subproblem (B_x^0, B_x^1, B_x^2, M) of x is equivalent to subproblem $(B_x^0, B_x^1, B_x^2 \cup \{v\}, M)$ of y.



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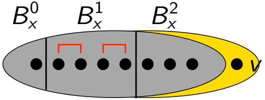

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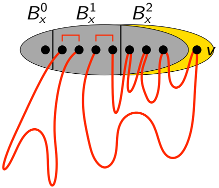
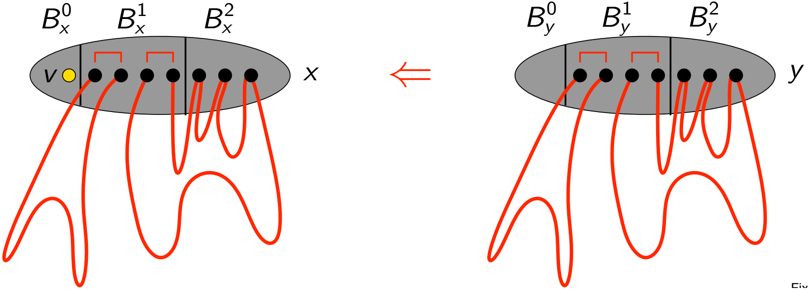


Hamiltonian cycle and nice tree decompositions

Solving subproblem (B_x^0, B_x^1, B_x^2, M) of node x.

Introduce: 1 child y, $B_x = B_y \cup \{v\}$ for some vertex v

Case 1: $v \in B_x^0$.
Subproblem is equivalent with $(B_x^0 \setminus \{v\}, B_x^1, B_x^2, M)$ for node y.



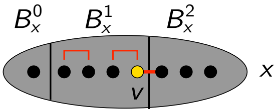
Hamiltonian cycle and nice tree decompositions

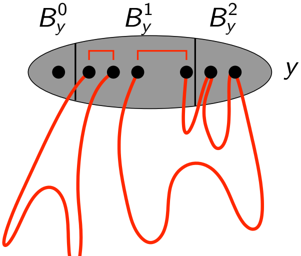
Solving subproblem (B_x^0, B_x^1, B_x^2, M) of node x.

Introduce: 1 child y, $B_x = B_y \cup \{v\}$ for some vertex v

Case 2:
$v \in B_x^1$. Every neighbor of v in V_x is in B_x. Thus v has to be adjacent with one other vertex of B_x.

Is there a subproblem $(B_y^0, B_y^1, B_y^2, M')$ of node y such that making a vertex of B_y adjacent to v makes it a solution for subproblem (B_x^0, B_x^1, B_x^2, M) of node x?





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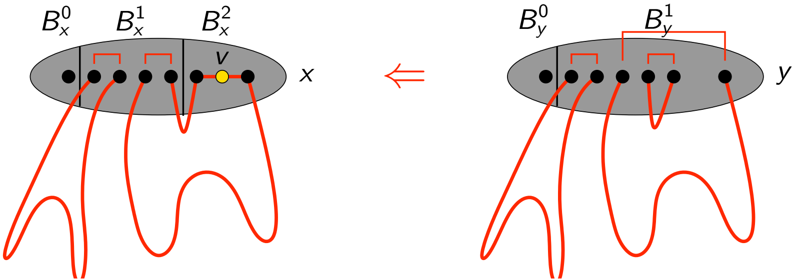
Case 3: $v \in B_x^1$. Similar to Case 2, but 2 vertices of B_y are adjacent with v.



Hamiltonian cycle and nice tree decompositions

Solving subproblem (B_x^0, B_x^1, B_x^2, M) of node x.

Join: 2 children y_1, y_2 with $B_x = B_{y_1} = B_{y_2}$

A solution H is the union of a subgraph $H_1 \subseteq G[V_{y_1}]$ and a subgraph $H_2 \subseteq G[V_{y_2}]$.

If H_1 is a solution for $(B_{y_1}^0, B_{y_1}^1, B_{y_1}^2, M_1)$ of node y_1 and H_2 is a solution for $(B_{y_2}^0, B_{y_2}^1, B_{y_2}^2, M_2)$ of node y_2, then we can check if $H_1 \cup H_2$ is a solution for (B_x^0, B_x^1, B_x^2, M) of node x.

For any two subproblems of y_1 and y_2, we check if they have solutions and if their union is a solution for (B_x^0, B_x^1, B_x^2, M) of node x.

Monadic Second Order Logic

Extended Monadic Second Order Logic (EMSO)

A logical language on graphs consisting of the following:

- Logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $=$, $\neq$
- quantifiers $\forall$, $\exists$ over vertex/edge variables
- predicate $\mathrm{adj}(u, v)$: vertices u and v are adjacent
- predicate $\mathrm{inc}(e, v)$: edge e is incident to vertex v
- quantifiers $\forall$, $\exists$ over vertex/edge set variables
- $\in$, $\subseteq$ for vertex/edge sets

Example: The formula
$\exists C \subseteq V \, \forall v \in C \; \exists u_1, u_2 \in C (u_1 \neq u_2 \wedge \mathrm{adj}(u_1, v) \wedge \mathrm{adj}(u_2, v))$ is true ...

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$\exists C \subseteq V \, \forall v \in C \; \exists u_1, u_2 \in C(u_1 \neq u_2 \wedge \mathrm{adj}(u_1, v) \wedge \mathrm{adj}(u_2, v))$ is true if graph $G(V, E)$ has a cycle.

Courcelle's Theorem

Courcelle's Theorem: If a graph property can be expressed in EMSO, then for every fixed $w \geq 1$, there is a linear-time algorithm for testing this property on graphs having treewidth at most w.

Note: The constant depending on w can be very large (double, triple exponential etc.), therefore a direct dynamic programming algorithm can be more efficient.

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If we can express a property in EMSO, then we immediately get that testing this property is FPT parameterized by the treewidth w of the input graph.

Can we express 3-Coloring and Hamiltonian Cycle in EMSO?

Using Courcelle's Theorem

3-Coloring

$$\exists C_1, C_2, C_3 \subseteq V \left(\forall v \in V\, (v \in C_1 \vee v \in C_2 \vee v \in C_3)\right) \wedge \left(\forall u, v \in V\, \mathrm{adj}(u, v) \to (\neg(u \in C_1 \wedge v \in C_1) \wedge \neg(u \in C_2 \wedge v \in C_2) \wedge \neg(u \in C_3 \wedge v \in C_3))\right)$$

Using Courcelle's Theorem

3-Coloring Hamiltonian Cycle

$\exists H \subseteq E\big(\text{spanning}(H) \wedge (\forall v \in V\, \text{degree2}(H, v))\big)$

$\text{degree0}(H, v) := \neg \exists e \in H\, \text{inc}(e, v)$

$\text{degree1}(H, v) := \neg \text{degree0}(H, v) \wedge \big(\neg \exists e_1, e_2 \in H\, (e_1 \neq e_2 \wedge \text{inc}(e_1, v) \wedge \text{inc}(e_2, v))\big)$

$\text{degree2}(H, v) := \neg \text{degree0}(H, v) \wedge \neg \text{degree1}(H, v) \wedge \big(\neg \exists e_1, e_2, e_3 \in H\, (e_1 \neq e_2 \wedge e_2 \neq e_3 \wedge e_1 \neq e_3 \wedge \text{inc}(e_1, v) \wedge \text{inc}(e_2, v) \wedge \text{inc}(e_3, v)))\big)$

$\text{spanning}(H) := \forall u, v \in V\, \exists P \subseteq H\, \forall x \in V\, \big(((x = u \vee x = v) \wedge \text{degree1}(P, x)) \vee (x \neq u \wedge x \neq v \wedge (\text{degree0}(P, x) \vee \text{degree2}(P, x)))\big)$

Using Courcelle's Theorem

Two ways of using Courcelle's Theorem:

1. The problem can be described by a single formula (e.g, 3-Coloring, Hamiltonian Cycle).

$\Rightarrow$ Problem can be solved in time $f(w) \cdot n$ for graphs of treewidth at most w.

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2. The problem can be described by a formula for each value of the parameter k.

Example: For each k, having a cycle of length exactly k can be expressed as

$$\exists v_1, \ldots, v_k \in V \, (\mathrm{adj}(v_1, v_2) \wedge \mathrm{adj}(v_2, v_3) \wedge \cdots \wedge \mathrm{adj}(v_{k-1}, v_k) \wedge \mathrm{adj}(v_k, v_1)).$$

⇒ Problem can be solved in time $f(k, w) \cdot n$ for graphs of treewidth w.

⇒ Problem is FPT parameterized with combined parameter k and treewidth w.

Subgraph Isomorphism

Subgraph Isomorphism: given graphs H and G, find a copy of H in G as subgraph. Parameter $k := |V(H)|$ (size of the small graph).

For each H, we can construct a formula ϕ_H that expresses "G has a subgraph isomorphic to H" (similarly to the k-cycle on the previous slide).

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$\Rightarrow$ By Courcelle's Theorem, Subgraph Isomorphism can be solved in time $f(H, w) \cdot n$ if G has treewidth at most w.

$\Rightarrow$ Since there is only a finite number of simple graphs on k vertices, Subgraph Isomorphism can be solved in time $f(k, w) \cdot n$ if H has k vertices and G has treewidth at most w.

$\Rightarrow$ Subgraph Isomorphism is FPT parameterized by combined parameter $k := |V(H)|$ and the treewidth w of G.

Part II:
Graph-theoretical properties of treewidth

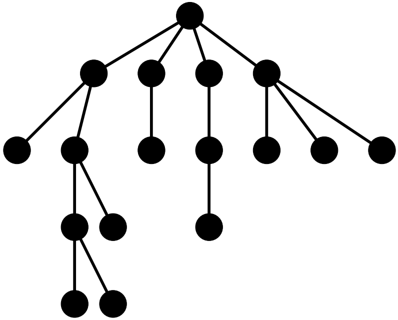
The Robber and Cops game

Game: k cops try to capture a robber in the graph.

- In each step, the cops can move from vertex to vertex arbitrarily with helicopters.
- The robber moves infinitely fast on the edges, and sees where the cops will land.

The Robber and Cops game (cont.)

Example: 2 cops have a winning strategy in a tree.



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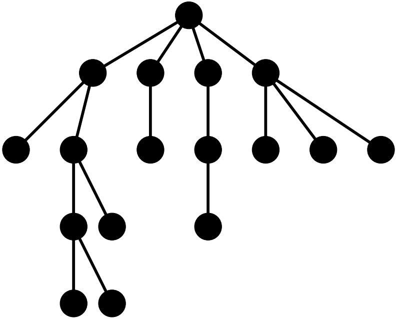

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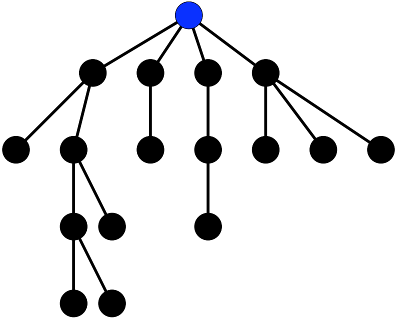

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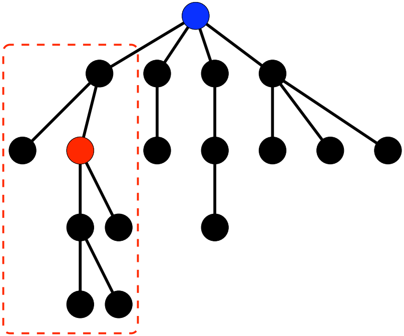

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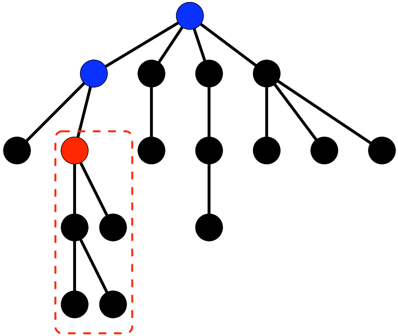

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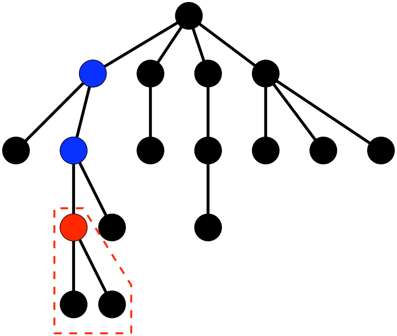

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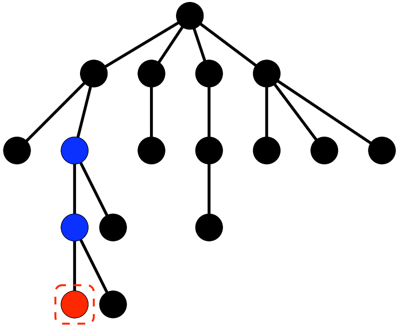

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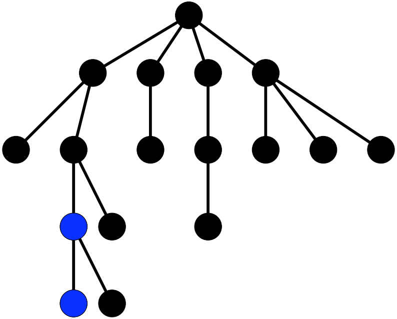

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$k+1$ cops can win the game iff the treewidth of the graph is at most k.

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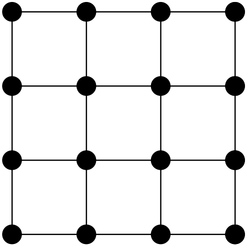
$\Downarrow$

For every fixed k, it can be checked in polynomial-time if treewidth is at most k.

Exercise 1: Show that the treewidth of the $k \times k$ grid is at least $k - 1$.
Exercise 2: Show that the treewidth of the $k \times k$ grid is at least k.

Properties of treewidth

Fact: For every $k \geq 2$, the treewidth of the $k \times k$ grid is exactly k.



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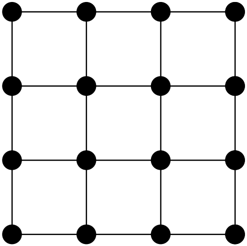


Fact: Treewidth does not increase if we delete edges, delete vertices, or contract edges.

$\Rightarrow$ If F is a **minor** of G, then the treewidth of F is at most the treewidth of G.

Excluded Grid Theorem

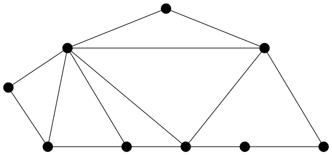
Fact: [Excluded Grid Theorem] If the treewidth of G is at least $k^{4k^2(k+2)}$, then G has a $k \times k$ grid minor.

A large grid minor is a "witness" that treewidth is large.

Fact: Every **planar graph** with treewidth at least $4k$ has $k \times k$ grid minor.

Outerplanar graphs

Definition: A planar graph is **outerplanar** if it has a planar embedding where every vertex is on the infinite face.

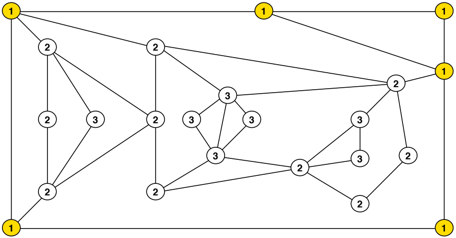


Fact: Every outerplanar graph has treewidth at most 2.

k-outerplanar graphs

Given a planar embedding, we can define **layers** by iteratively removing the vertices on the infinite face.

Definition: A planar graph is k-**outerplanar** if it has a planar embedding having at most k layers.



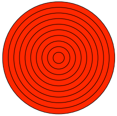
Fact: Every k-outerplanar graph has treewidth at most $3k + 1$.

Part III:
Applications

Baker's shifting strategy

Subgraph Isomorphism for planar graphs: given planar graphs H and G, find a copy of H in G as subgraph. Parameter $k := |V(H)|$.

Layers of the planar graph: (as in the definition of k-outerplanar):



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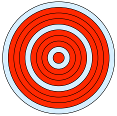

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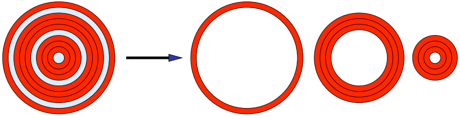

- For a fixed $0 \leq s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$
- The resulting graph is k-outerplanar, hence it has treewidth at most $3k+1$.
- Using the $f(k,w) \cdot n$ time algorithm for Subgraph Isomorphism, the problem can be solved in time $f(k, 3k+1) \cdot n$.

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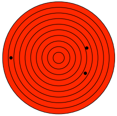

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- We do this for every $0 \leq s < k+1$: for at least one value of s, we do not delete any of the k vertices of the solution $\Rightarrow$ we find a copy of H in G if there is one.
- Subgraph Isomorphism for planar graphs is FPT parameterized by $k := |V(H)|$.

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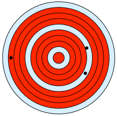

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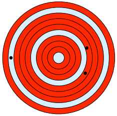

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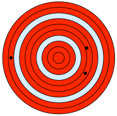
- Using the $f(k,w) \cdot n$ time algorithm for Subgraph Isomorphism, the problem can be solved in time $f(k, 3k+1) \cdot n$.
- We do this for every $0 \leq s < k+1$: for at least one value of s, we do not delete any of the k vertices of the solution $\Rightarrow$ we find a copy of H in G if there is one.
- Subgraph Isomorphism for planar graphs is FPT parameterized by $k := |V(H)|$.

Baker's shifting strategy

Subgraph Isomorphism for planar graphs: given planar graphs H and G, find a copy of H in G as subgraph. Parameter $k := |V(H)|$.

Layers of the planar graph: (as in the definition of k-outerplanar):



- For a fixed $0 \leq s < k+1$, delete every layer L_i with $i = s \pmod{k+1}$
- The resulting graph is k-outerplanar, hence it has treewidth at most $3k+1$.
- Using the $f(k,w) \cdot n$ time algorithm for Subgraph Isomorphism, the problem can be solved in time $f(k, 3k+1) \cdot n$.
- We do this for every $0 \leq s < k+1$: for at least one value of s, we do not delete any of the k vertices of the solution $\Rightarrow$ we find a copy of H in G if there is one.
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Detour to approximation. . .

Detour to approximation algorithms

Definition: A c-**approximation** algorithm for a maximization problem is a polynomial-time algorithm that finds a solution of cost at least OPT/c.

Definition: A c-**approximation** algorithm for a minimization problem is a polynomial-time algorithm that finds a solution of cost at most $\mathrm{OPT} \cdot c$.

There are well-known approximation algorithms for NP-hard problems: $\frac{3}{2}$-approximation for Metric TSP, 2-approximation for Vertex Cover, $\frac{8}{7}$-approximation for Max 3SAT, etc.

- For some problems, we have lower bounds: there is no $(2-\epsilon)$-approximation for Vertex Cover or $(\frac{8}{7}-\epsilon)$-approximation for Max 3SAT (under suitable complexity assumptions).
- For some other problems, arbitrarily good approximation is possible in polynomial time: for any $c > 1$ (say, $c = 1.000001$), there is a polynomial-time c-approximation

Approximation schemes

Definition: A **polynomial-time approximation scheme (PTAS)** for a problem P is an algorithm that takes an instance of P and a rational number $\epsilon > 0$,

- always finds a $(1+\epsilon)$-approximate solution,
- the running time is polynomial in n for every fixed $\epsilon > 0$.

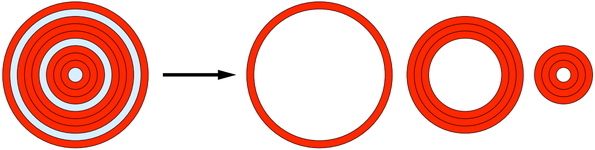
Typical running times: $2^{1/\epsilon} \cdot n$, $n^{1/\epsilon}$, $(n/\epsilon)^2$, n^{1/ϵ^2}.

Some classical problems that have a PTAS:

- Independent Set for planar graphs
- TSP in the Euclidean plane
- Steiner Tree in planar graphs
- Knapsack

Baker's shifting strategy for EPTAS

Fact: There is a $2^{O(1/\epsilon)} \cdot n$ time PTAS for INDEPENDENT SET for planar graphs.



- Let $D := 1/\epsilon$. For a fixed $0 \leq s < D$, delete every layer L_i with $i = s \pmod{D}$
- The resulting graph is D-outerplanar, hence it has treewidth at most $3D + 1 = O(1/\epsilon)$.
- Using the $O(2^w \cdot n)$ time algorithm for INDEPENDENT SET, the problem can be solved in time $2^{O(1/\epsilon)} \cdot n$.
- We do this for every $0 \leq s < D$: for at least one value of s, we delete at most $1/D = \epsilon$ fraction of the solution $\Rightarrow$ we get a $(1 + \epsilon)$-approximate solution.

Back to FPT. . .

Bidimensionality

A powerful framework to obtain efficient algorithms on planar graphs.

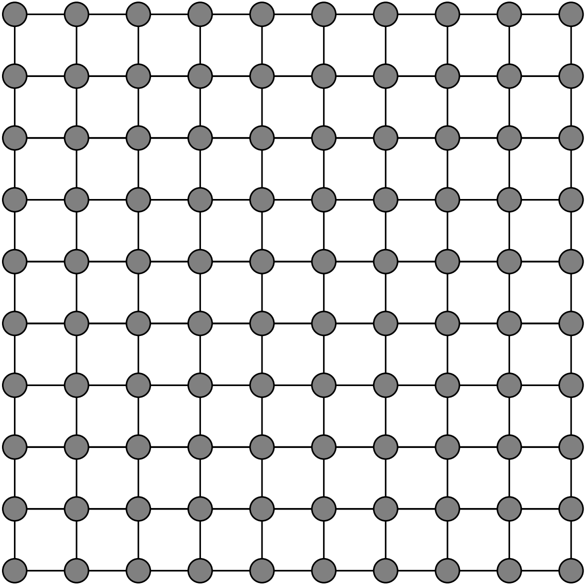
Let $x(G)$ be some graph invariant (i.e., an integer associated with each graph).
Some typical examples:

- Maximum independent set size.
- Minimum vertex cover size.
- Length of the longest path.
- Minimum dominating set size
- Minimum feedback vertex set size.

Given G and k, we want to decide if $x(G) \le k$ (or $x(G) \ge k$).

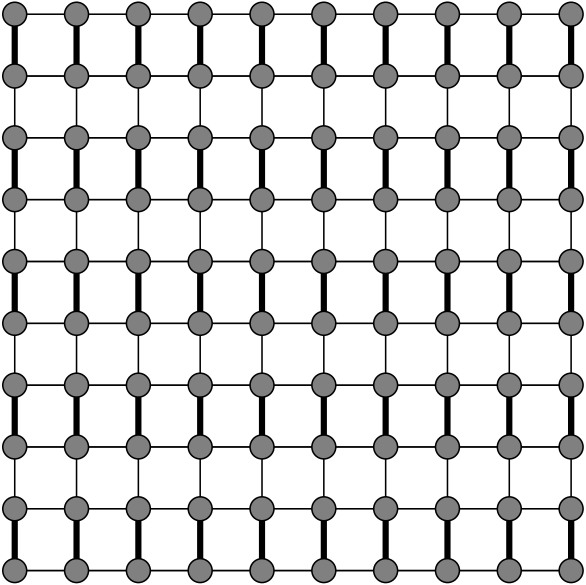
For many natural invariants, we can do this in time $2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Bidimensionality for Vertex Cover



$H_{r,r}$ for $r = 10$

Bidimensionality for Vertex Cover



$\mathbf{vc}(H_{r,r}) \geq \frac{r^2}{2}$

Bidimensionality for VERTEX COVER

Observation: If the treewidth of a planar graph G is at least $4\sqrt{2k}$
$\Rightarrow$ It contains a $\sqrt{2k} \times \sqrt{2k}$ grid minor (Excluded Grid Theorem for planar graphs)
$\Rightarrow$ The vertex cover size of the grid is at least k in the grid
$\Rightarrow$ Vertex cover size is at least k in G.

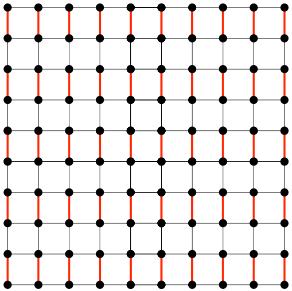
We use this observation to solve VERTEX COVER on planar graphs as follows:

- Set $w := 4\sqrt{2k}$.
- Use the 4-approximation tree decomposition algorithm ($2^{O(w)} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$ time).
  - If treewidth is at least w: we answer 'vertex cover is $\geq k$'.
  - If we get a tree decomposition of width $4w$, then we can solve the problem in time $2^{w} \cdot n^{O(1)} = 2^{O(\sqrt{k})} \cdot n^{O(1)}$.

Bidimensionality (cont.)

Definition: A graph invariant $x(G)$ is **minor-bidimensional** if

- $x(G') \leq x(G)$ for every minor G' of G, and
- If G_k is the $k \times k$ grid, then $x(G_k) \geq ck^2$ (for some constant $c > 0$).



Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

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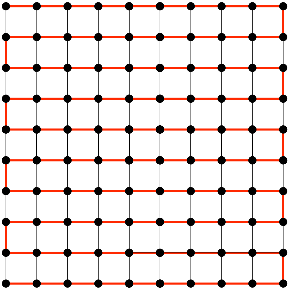

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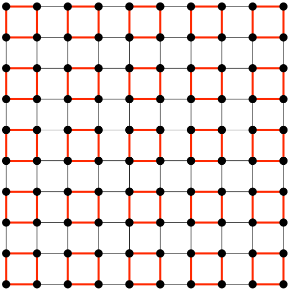

Examples: minimum vertex cover, length of the longest path, feedback vertex set are minor-bidimensional.

Bidimensionality (cont.)

We can answer "$x(G) \geq k$?" for a minor-bidimensional parameter the following way:

- Set $w := c\sqrt{k}$ for an appropriate constant c.
- Use the 4-approximation tree decomposition algorithm.
  - If treewidth is at least w: $x(G)$ is at least k.
  - If we get a tree decomposition of width $4w$, then we can solve the problem using dynamic programming on the tree decomposition.

Running time:

- If we can solve the problem using a width w tree decomposition in time $2^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k})} \cdot n^{O(1)}$.
- If we can solve the problem using a width w tree decomposition in time $w^{O(w)} \cdot n^{O(1)}$, then the running time is $2^{O(\sqrt{k} \log k)} \cdot n^{O(1)}$.

Summary

- Notion of treewidth allows us to generalize dynamic programming on trees to more general graphs.
- Standard techniques for designing algorithms on bounded treewidth graphs: dynamic programming and Courcelle's Theorem.
- Surprising uses of treewidth in other contexts (planar graphs).